On the filter approach to perceptual transparency

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In F. Faul and V. Ekroll (2002), we proposed a filter model of perceptual transparency that describes typical color changes caused by optical filters and accurately predicts perceived transparency. Here, we provide a more elaborate analysis of this model: (A) We address the question of how the model parameters can be estimated in a robust way. (B) We show that the parameters of the original model, which are closely related to physical properties, can be transformed into the alternative parameters hue H, saturation S, transmittance V, and clarity C that better reflect perceptual dimensions of perceived transparency. (C) We investigate the relation of H, S, V, and C to the physical parameters of optical filters and show that C is closely related to the refractive index of the filter, whereas V and S are closely related to its thickness. We also demonstrate that the latter relationship can be used to estimate relative filter thickness from S and V. (D) We investigate restrictions on S that result from properties of color space and determine its distribution under realistic choices of physical parameters. (E) We experimentally determine iso-saturation curves that yield nominal saturation values for filters of different hue such that they appear equally saturated.

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Introduction

The first systematic empirical studies on perceptual transparency were undertaken by members of the Gestalt school at the beginning of the last century. This is not surprising, because perceptual transparency is a nice example for their central concept that the whole is different from the sum of the parts: The stimulus shown in the top left panel of Figure 1, for example, is not interpreted as a mosaic of 8 independent colored regions, but the circular region is instead seen as a homogeneous red transparent layer floating in front of an achromatic background. However, this transparency interpretation only occurs if the colors in the stimulus are properly related to each other (as demonstrated in the right panel of Figure 1). Correspondingly, one of the central goals of research on perceptual transparency is to identify these critical color relations and to describe how they determine the properties of the perceived transparent layer.

To solve this problem, one may start directly at the proximal stimulus. Viable strategies to identify relevant color relations are then, for example, to test which of a set of candidate transforms of the background colors elicits perceived transparency (D’Zmura, Colantoni, Knoblauch, & Laget, 1997) or to try to derive them from empirical data gathered in (matching) experiments (e.g., Anderson & Khang, 2010; Anderson, Singh, & Meng, 2006; Fulvio, Singh, & Maloney, 2006; Singh & Anderson, 2002; Wollschläger & Anderson, 2009). There are indeed many important aspects of transparency perception that can only be investigated in this way, for example, all phenomena that presumably reflect idiosyncratic strategies used in the visual system, such as grouping processes (Anderson et al., 2006; Koenderink, van Doorn, Pont, & Richards, 2008; Koenderink, van Doorn, Pont, & Wijntjes, 2010).

These direct approaches emphasize—in our view correctly—the importance of phenomenological criteria, but they neglect the potential heuristic value of exploring the functional role of vision. In this paper, we pursue a research strategy that takes this into account by considering transparency perception as a detection task. It is based on the observation that it is, in general, possible to reliably infer the presence and properties of external objects from patterns in the proximal stimulus, which implies that each visually distinguishable object is characterized by a specific signature in the input that can be used as a cue in the inference process. To identify “signatures” that may be relevant in transparency perception, this approach starts from objects that usually appear transparent and then analyzes the image generation process that “writes” the proximal stimulus. The goal is to find formal models that describe regularities induced into the input by such objects. Whether a model found in this way describes regularities that are actually used by the visual system as a cue must then be tested empirically.

There exist two lines of research that both use this strategy but refer to object classes with different image generation processes. The prototypical object in the "episcotister approach" (Da Pos, 1989; Faul, 1996; Gerbino, Stultiens, Troost, & de Weert, 1990; Kasrai & Kingdom, 2001; Koenderink et al., 2008; Metelli, 1970;
Richards, Koenderink, & Doorn, 2009) is a sector disk (the “episcotister”) that swiftly rotates in front of a nonuniform background. The relevant objects in the “filter approach” (Beck, 1978; Beck, Prazdny, & Ivry, 1984; Faul & Ekroll, 2002; Khang & Zaidi, 2002; Nakauchi, Silfsten, Parkkinen, & Usui, 1999; Robilotto & Zaidi, 2004; Westland & Ripamonti, 2000) are “optical filters,” like (colored) glass, fluids, and plastics. In the following, we will focus on transparent objects, that is, clear filters. A more general case are translucent objects, e.g., milky glass, which involve light scattering (Brill, 1984; Koenderink et al., 2008).

Although most objects we perceive as transparent belong to the optical filters class, research based on the episcotister approach clearly dominates in the literature. The episcotister approach appears attractive not only because it is very simple but also because it seems to be in fairly good accord with experimental findings, especially in the achromatic domain (see, e.g., Kasrai & Kingdom, 2001). That it often makes rather good predictions is also documented by the routine use of the corresponding alpha-blending technique in computer graphics to render transparent overlays. The simplicity of the episcotister model stems in part from the fact that it involves only a form of additive color mixture (partitive color mixture) so that information about the spectral properties of the scene, which cannot be recovered based on the retinal signal, are irrelevant. The filter approach, on the other hand, may seem much less tractable, because the image generation process associated with “optical filters” is highly complex and, in particular, involves subtractive color mixture, where spectral information plays a crucial role.

A further methodological advantage is that the filter approach suggests hypotheses concerning the perception of transparency in more natural situations than those traditionally studied. This heuristic value of the filter approach is a positive aspect of the complexity of the image generation process: The images cast by optical filters depend in complex ways on the illumination, the reflectance of background surfaces, the properties of the filter, and the viewing geometry. The rich structure of the model makes it possible to generate a much wider range of qualitatively different input images than is possible with the episcotister model (the image cast by a clear plastic bottle, for instance, has no natural place in the latter model). Furthermore, it also poses strong restrictions on how the retinal image changes with variations in the parameters of the physical model. It predicts, for instance, how the strength of the mirror image of the environment that is normally reflected from

![Figure 1](http://jov.arvojournals.org/) The circular region in the stimulus appears transparent if the colors are chosen properly (left) but opaque if not (right).
the surface of optical filters (see Figure 2) varies with changes in viewing angle or how the color of the transmitted light changes with the thickness of the optical filter. Thus, one possible way to gain insights into the kind of internal representations the visual system uses in recognizing such objects is to test to what extent violations of corresponding regularities hamper the transparency impression.

The filter approach also reveals the close relationship of research on perceived transparency to other research questions: With research on color constancy (e.g., Land, 1959; Maloney, 1999; Maloney & Wandell, 1986; Smithson, 2005), it shares the problems arising from subtractive color mixtures, and with research on gloss perception (e.g., Beck & Prazdny, 1981; Dove, 1850; Fleming, Dror, & Adelson, 2003; Wendt, Faul, Ekroll, & Mausfeld, 2010), it shares the problem of how to isolate mirror reflections of the environment from material-related contributions to the reflected light. This makes it possible to capitalize on advances in these related research domains.

Thus, the filter approach is not only a viable alternative that (at least under some conditions) leads to better predictions than the episcotister approach, but it also constitutes a conceptual framework that suggests a great number of different testable hypotheses and provides heuristic guidance for embarking on the study of more complex natural situations.

As part of a more comprehensive research program suggested by the filter approach, the present investigations represent some first steps in exploring its viability and fruitfulness. We start from the filter model proposed in Faul and Ekroll (2002). After a brief review of relevant aspects of this model in the Filter transparency in simple stimuli section, we explore it more closely in several areas related to the distal and phenomenological criteria mentioned above.

In the Robust parameter estimation section, we investigate how the parameters of the model can be estimated from the input in a robust way. In this context, we also consider the case of more than two background colors and cases where some regions are completely covered by the filter.

In the Comparison of submodels of the filter model section, we experimentally compare a full filter model with a simpler submodel that we used in Faul and Ekroll (2002) in order to establish which of them is best suited as a point of departure for subsequent investigations. This question was raised by the observation that the former model predicts color changes of optical filters clearly more accurately, whereas the latter seemed to predict perceived transparency slightly better.

In the Remapping the model parameters to a phenomenological space section, we propose a remapping of the parameters of the original model that were inspired by properties of the physical model to the alternative dimensions hue, saturation, transmittance, and clarity that better describe the phenomenal impression of the transparent layer.

In The relation of model parameters to the physical model section, we use computer simulations to investigate how these “subjective” parameters relate to physical
parameters of optical filters. Our results indicate that the saturation and transmittance parameters depend in a regular way on the thickness of the optical filter, and that the clarity parameter depends on the refractive index of the filter material. We also demonstrate that the relationship between the transmittance parameter and filter thickness is close enough to allow a rather accurate estimate of the relative thickness of the filter from the spatial transmittance distribution. The same holds for the saturation parameter.

In the Restrictions on layer saturation section, Distribution of filter parameters section, and Iso-saturation curves section, we investigate several questions regarding the restrictions and meaning of the saturation parameter: In the Restrictions on layer saturation section, we first explore restrictions on filter saturation that are imposed by properties of color space. In the Distribution of filter parameters section, we use computer simulations to gain insights into the distribution of the filter parameters for realistic choices of scene parameters. Finally, in the Iso-saturation curves section, we report results of an experiment aimed to determine iso-saturation curves that describe the nominal saturation values across different hues that lead to filters of equal perceived saturation.

Due to the relatively large number of topics dealt with in the text, it may sometimes be difficult to see their inner coherence. It may be helpful to first read the summary in the Summary and general discussion section in which we recapitulate central findings and clarify their relationship.

Filter transparency in simple stimuli

In Faul and Ekroll (2002), we analyzed the image generation process of optical filters under a number of simplifying assumptions: (A) all spectra are smooth and relatively broadband, (B) the illumination is diffuse and spatially uniform, (C) the filter is flat and of constant thickness, (D) filter and background are coplanar, and (E) the viewing direction is perpendicular to the filter surface.

With the exception of assumption (A), which presumably reflects true natural regularities, these assumptions basically ensure that the resulting images are similar to the simple stimuli traditionally used in experimental tests of transparency models. This is illustrated by the stimulus shown in the top left panel of Figure 2, which was computed by simulating the image generation process under restrictions (B)–(E) with a physically correct 3D rendering software (“mental ray” from NVIDIA). The other panels in Figure 2 show how the images of the same filter and background change if some of these restrictions are lifted. In the bottom left image, the filter is no longer coplanar to the background and the viewing direction has also changed; thus, (D) and (E) do not apply. The two panels in the middle column illustrate how the images shown in the left panels change if the illumination is no longer uniform as assumed in (B), and in the two rightmost panels also restriction (C) of a flat filter does no longer hold. In the middle and right panels, an image-based lighting technique (Debevec, 1998) was used to simulate the illumination inside an airport terminal (which can partly be seen in the background of the lower panels). Using a nonuniform illumination makes it obvious that the top surface of an optical filter reflects a mirror image of the environment that is spatially distorted if the surface is not flat. Another interesting side effect is that the filter surface now looks more or less glossy, whereas it has a dull appearance under a uniform diffuse illumination.

These examples, especially those in the top row of Figure 2, also demonstrate that the resulting images under more complex conditions are often remarkable similar to those resulting under the strong restrictions made in deriving the model. This makes it plausible that regularities found in the simple stimuli are also useful in more complex cases.

Image generation

If all of the above restrictions are applied, then the image generation model is relatively simple and can be given in closed form. We briefly describe the model here, because we will often refer to its properties and it will also be used in the simulations reported in the Comparison of submodels of the filter model section, The relation of model parameters to the physical model section, and Distribution of filter parameters section. A more detailed exposition can be found in Faul and Ekroll (2002).

The three parameters of the filter are the absorption spectrum \( m(\lambda) \), \( 0 \leq m(\lambda) \leq 1 \), the filter thickness \( s \geq 0 \), and the refractive index \( n(\lambda) \). The latter is assumed to be a constant function of wavelength, with \( n \geq 1 \). The Bouguer–Beer law, \( I/I_0 = \theta(\lambda) = \exp[-m(\lambda)x] \), describes how the inner transmittance \( \theta(\lambda) \) (the ratio of the amount \( I_1 \) of light reaching the bottom of the filter to the amount \( I_0 \) entering at the top) depends on absorption and thickness. Fresnel’s equations describe how the relative amount \( k \) of light that is specularly reflected at an air–filter interface depends on the angle of the incoming light and the refractive index. For normal incidence as assumed here, \( k = \frac{(n-1)}{(n+1)^2} \).

For a flat filter of constant thickness and normal incidence of light, the total reflection \( r(\lambda) \) and total transmittance \( t(\lambda) \) (i.e., the relative amounts of light leaving the filter after multiple inner reflections at the illuminated and opposite sides, respectively) can be given in closed form: \( r(\lambda) = k + k(1-k)\theta^2(\lambda)/[1-k^2\theta^2(\lambda)] \), and \( t(\lambda) = (1-k)\theta(\lambda)/[1-k^2\theta^2(\lambda)] \). If the filter is placed in front of a background with reflectance \( a(\lambda) \), then the virtual reflectance \( p(\lambda) \) of the filter surface (i.e., the
relative amount of incident light that is reflected from the filter area) can be written as

\[ p(\lambda) = \frac{I^2(\lambda) a(\lambda)}{1 - r(\lambda) a(\lambda)} + r(\lambda). \] (1)

Given \( p(\lambda) \), the cone excitation \( P_i, i = L, M, S \), can be computed in the usual way, that is, \( P_i = \int p(\lambda) I(\lambda) R_i(\lambda) d\lambda \), where \( I(\lambda) \) is the illumination spectrum and \( R_i(\lambda) \) is the sensitivity spectrum of cone class \( i \).

**Psychophysical model**

In Faul and Ekroll (2002), we showed that the color changes caused by optical filters can be well described by a simple model formulated in terms of color codes. The predictions of this psychophysical filter model for four-color stimuli, like the one shown in Figure 1, are given by

\[ P_i = \tau_i(A_i + \delta I_i) + \mu \delta I_i, \] (2)

\[ Q_i = \tau_i(B_i + \delta I_i) + \mu \delta I_i, \] (3)

where \( A \) and \( B \) denote the color codes of the bipartite background region, and \( P \) and \( Q \) denote the color codes of the same regions viewed through the filter (see Equations 17 and 18 in Faul & Ekroll, 2002). In the following, we will always assume that the color codes are cone excitation values, where the index indicates one of \( L, M, S \).

The four colors \( A, B, P, Q \) are the observables that provide the basis for inferring transparency. The remaining variables are parameters of the model that are not directly given in the input: \( I \) is the color of the illumination, \( \tau \) is the vector of transmittance factors, and \( \delta \) is a factor related to the amount of direct reflection. A comparison of the model equations with a simplified version of the image generation model suggests (see Faul & Ekroll, 2002, p. 1086) that \( \tau \) roughly represents the squared total transmittance \( \tau^2(\lambda) \) and that \( \delta \) is related to the direct reflection factor \( k \). This motivates the parameter restrictions \( 0 \leq \tau_i \leq 1 \) and \( \delta \geq 0 \). The remaining parameter \( \mu \) controls the relative amount of directly reflected light of first order that is reflected from the top surface of the filter to higher order contributions that traveled through the filter and are, thus, affected by its transmissive properties. Here, it mainly has a technical meaning and is used to distinguish between different submodels. The model in which \( \mu = 1 \) will, in the following, be called the **full model** (see Equations 17 and 18 in Faul & Ekroll, 2002).

From the general model in Equations 2 and 3, more specific ones can be derived by specifying a certain method to estimate \( I \) and/or by setting different values for \( \mu \). For all models of this class, it holds that

\[ \tau_i = \frac{P_i - Q_i}{A_i - B_i}. \] (4)

This shows that the transmittance factors do not depend on the illumination. The computation of the factor \( \delta \), in contrast, requires knowledge about the illumination. For the general model, we have

\[ \delta = \frac{A_i Q_i - B_i P_i}{(A_i - B_i + P_i - Q_i) I_i}. \] (5)

In Faul and Ekroll (2002), we focused on the special case of the above model with \( \mu = 0 \):

\[ P_i = \tau_i(A_i + \delta I_i), \] (6)

\[ Q_i = \tau_i(B_i + \delta I_i). \] (7)

In this **reduced model**, the equation for \( \delta \) simplifies to

\[ \delta = \frac{A_i Q_i - B_i P_i}{(P_i - Q_i) I_i}. \] (8)

The further assumption that the illumination is estimated from the arithmetic mean of the background color, that is, \( I = (A + B) / 2 \), finally leads to

\[ \delta = \frac{2(A_i Q_i - B_i P_i)}{(A_i + B_i)(P_i - Q_i)}, \] (9)

an equation for \( \delta \) containing also only observables.

**Robust parameter estimation**

The formulae given above demonstrate that the visual array contains enough information to estimate the unknowns of the filter model. There are, however, several reasons why using them in the given form is probably not the best way to do the actual estimation. First, the formula for \( \tau_i \) and \( \delta \) have singularities at \( A_i = B_i \) and \( P_i = Q_i \), respectively. Thus, even if the model describes the stimulus exactly, the computation may be undefined or may at least become rather unstable if the background or the filtered colors are similar within one or more color
channels. Second, the computations do not consider the presence of noise that must be assumed under realistic viewing conditions. Third, the computation uses only local information between four colors (for example, at X-junctions) and ignores more global information that may improve the accuracy of the computation in more complex stimuli. This local computation also requires the exact localization and assignment of the four related colors, which may be difficult to achieve if the background has a fine or complex texture. The latter aspect is especially problematic, because due to refraction effects exact alignment of contours on both sides of the border of an optical filter cannot usually be expected.

**Estimation procedures**

We will now discuss algorithms that alleviate most of the above-mentioned problems.

**Case 1: \( N = 2 \) and \( I = \text{mean (background)} \).** We first consider the special case of two background colors and \( I = (A + B)/2 \), because it is especially easy and highlights some important regularities: Adding Equations 6 and 7 for the two filtered colors \( P \) and \( Q \) yields \( P_i + Q_i = \tau_j(A_i + B_j + \delta(A_i + B_j)) \). A simple transformation of this equation leads to

\[
\frac{1}{1 + \delta} = \frac{(A_i + B_i)}{(P_i + Q_i)}\tau_i. \tag{10}
\]

We define \( R_i := (P_i + Q_i)/(A_i + B_i) \) and observe that this quantity—a ratio of means—can be computed in a robust way: The only (practically irrelevant) singularity occurs at \( A_i = B_i = 0 \), that is, if the background is completely black. We further define \( \gamma := 1/(1 + \delta) \). This provides a convenient reparameterization that maps the infinite parameter range \([0, \infty]\) of the “direct reflection factor” \( \delta \) onto the finite interval \([0, 1]\), where \( \gamma = 1 \) for \( \delta = 0 \) and where \( \gamma \) approaches zero for \( \delta \rightarrow \infty \). As will be discussed in the Remapping the model parameters to a phenomenological space section, \( \gamma \) bears a close relationship to the perceived clarity of the filter.

With these definitions, we have \( \gamma R_i = \tau_i \). To estimate \( \tau \) in a robust way, we first compute \( \tau_i = (P_i - Q_i)/(A_i - B_i) \) for channel \( i \) in which the contrast \( |A_i - B_i| \) is maximal, then compute \( \gamma = \tau_i/R_i \) and finally set \( \tau = \gamma R \). Figure 3 gives a visual representation of the solution using the above definitions. Note that \( \gamma \) and \( R_i \) are not independent, but that \( R_i > 1 \) (a brightening effect) implies \( \gamma < 1 \).

**Case 2: \( N \geq 2 \) colors and \( I = \text{mean (background)} \).** If \( N \geq 2 \) background/filter color pairs \((A_i', P_i')\) are given, then under the assumption \( I_i = \text{mean}(A_i) = 1/N \sum_i A_i' \) we get analogous to Case 1:

\[
\gamma = 1/(1 + \delta) = \frac{\text{mean}(A_i)}{\text{mean}(P_i)}\tau_i. \tag{11}
\]

Figure 3. Visualization of the solution in Case 1. The three colored lines with slope \( 1/R_i \) describe the filter effect in color channel \( i \). Line inside the shaded region \((R_i < 1)\) indicates a darkening effect, whereas a line outside the shaded region \((R_i > 1)\) indicates a brightening effect. A clarity value \( \gamma \) is a valid solution if a horizontal line through this value intersects the three colored lines inside the unit area, and in that case, the projections of the intersection points on the horizontal axis are the corresponding transmittance values \( \tau_i \) (see text for more details).

Here, we define \( R_i := \text{mean}(A_i)/\text{mean}(P_i) \), so again \( \gamma R = \tau \). To compute \( \tau \) directly, we use \( \tau_i = \text{sd}(P_i)/\text{sd}(A_i) \), where \( \text{sd}(X_i) \) denotes the standard deviation of \( N \) colors \( X' \) in channel \( i \). It is easy to see that this is actually a solution for \( \tau \), because since \( P_i' = \tau_i A_i' + \tau_j I_j \) with \( C_i := \tau_i I_j \) constant, we have \( \var(P_i) = \var(\tau_i A_i + C_i) = \tau_i^2 \var(A_i) \), and thus, \( \tau_i = \text{sd}(P_i)/\text{sd}(A_i) \). This method uses information from all available colors and, thus, provides a more robust solution than computations based on (random) subsets of these colors.

This method has a singularity at \( \text{sd}(A_i) = 0 \), that is, \( \tau_i \) cannot be computed if the background does not vary in color channel \( i \). To compute \( \tau \) in a robust way, we may use a similar strategy as above, that is, we first compute \( \tau_i = \text{sd}(P_i)/\text{sd}(A_i) \) for channel \( i \) in which \( \text{sd}(A_i) \) is maximal, and then proceed in exactly the same way as before.

**Case 3: The general case.** We now consider the case of \( N \geq 2 \) background/filter color pairs \((A_i', P_i')\) and the general model given in Equations 2 and 3 with no special assumptions about how the illumination \( I \) is estimated from the input. A direct method to compute \( \tau \) is again \( \tau_i = \text{sd}(P_i)/\text{sd}(A_i) \). A different, indirect way is:
\( \mu \delta I_i = (\text{mean}(A_i) + \delta I_i) \). The latter method obviously requires that \( \delta \) and \( I \) are known, but it has the advantage that the computation is more robust. To compute \( \tau \) using this method, we again start by computing \( \tau_j = \text{std}(P_i^j) / \text{std}(A_i) \) for channel \( i \) in which \( \text{std}(A_i) \) is maximal. Using this value, we can compute \( \delta = u_i / v_i \), where \( u_i := \text{mean}(P_i) - \tau_i \text{mean}(A_i) \) and \( v_i := (\tau_i + \mu) I_i \). With \( \delta \) known, the indirect method is applied to compute \( \tau \) for the other color channels.

An alternative method (also applicable to Cases 1 and 2) is to estimate \( \delta_i \) in all channels, where \( \tau_i \) is defined (i.e., where \( \text{std}(A_i) > 0 \)) and to integrate them into a single estimate \( \delta \). In principle, any integration function \( f \) may be used, which gives the correct solution if the model describes the stimulus exactly. Thus, the choice of \( f \) only matters if the stimulus deviates from the model, and in this case, it influences the weighting of the model fluctuations across channels. Two integration functions have been found to work especially well, namely, \( f_M := \text{mean}(\delta_i) \) and \( f_L := (\sum \text{var}(u_i)) / (\sum v_i^2) \) with \( u_i \) and \( v_i \), as defined above. Simulations have shown that the model parameters estimated using these integration functions are very similar to those obtained with a least square fit, as described in the next paragraph.

**Iterative procedures.** Iterative (fit) procedures may also be used to estimate the parameters of a complex stimulus in a robust way. A simple method is to find the parameters \( \tau \) and \( \delta \) that minimize the loss function \( E = \sum_{i=1}^{N} \sum_{j=1}^{3} (P_i^j - \text{model})^2 / w_i \) under the constraints \( 0 \leq \tau_i \leq 0 \) and \( \delta \geq 0 \). Here, \( P_i^j \) is the model prediction for the observed \( P_i^j \) and \( w_i \) is a factor that compensates for different scales in the color channels (\( w \) may, for example, be chosen to be the cone excitations elicited by a constant light spectrum).

### Local vs. global procedures

In stimuli with complex backgrounds, estimation procedures can be more or less local. A completely local procedure uses only information at one \( X \)-junction, whereas global procedures may also use information from more distant parts of the retinal image. The statistical estimation that underlies “global estimates” can be done in a spatial or a temporal manner. In the first case, the procedure would be across responses of neighboring receptors getting input from different areas of the background and the filter region. In the latter case, the integration is across the responses of single receptors, for example, during a random walk across the stimulus caused by ocular tremor while fixating the stimulus.

A simple way to apply the global approach (using spatial integration) would be to use the strategy described under Case 2 for the complete stimulus. In this case, all colors are used at once to estimate \( I \), \( \tau \), and \( \delta \). Applying instead Case 1 locally at all \( m \) different \( X \)-junctions of the stimulus would yield \( m \) estimates of \( I \), \( \tau \), and \( \delta \), and an additional mechanism would be required to integrate these local estimates.

In most cases, local and global estimates are not identical, even if the model equation describes the stimulus exactly and the filter properties and the illumination are uniform. The reason is that local estimates of the illumination usually differ from global ones (which are, in general, more correct). The only exception to this rule is the parameter \( \tau \) if computed with the “direct procedure,” because this computation is illumination invariant. This inconsistency problem does not occur, however, if we estimate the illumination globally and then use the procedure described above as “general case” for the two background/filter color pairs at each \( X \)-junction.

The global approach has the additional advantage that it does not require the exact assignment of corresponding colors \( A_i \) and \( P_i \). Thus, it also works if there are nonmatching pairs of colors (for example, those belonging to regions that are completely covered by the filter or are only visible in plain view) or if it is difficult to assign corresponding \( A_i \), \( P_i \) pairs (for example, in complex backgrounds with a fine structure). The global method is, in general, also more robust, because the parameter estimation is based on more input. However, the advantages of the global approach do only apply if the actual situation does not change. That is, the filter properties, the illumination, and the background texture must be approximately constant within the integration region. It seems, therefore, advantageous to use adaptive areas of integration and/or—depending on the structure of the stimulus—different estimation strategies. Such adaptive estimation strategies might, for example, explain why contrast polarity at an \( X \)-junction, which has been found to be a very strong constraint in simple stimuli with large areas and few \( X \)-junctions, is considerably less important and may even be ignored if the structure of the background gets more complex (see Figure 19 in Singh & Anderson, 2002).

Adaptive strategies require criteria to decide whether to integrate or not. In this respect, the statistical estimation procedures considered above appear somewhat incomplete: They allow to correctly estimate the parameters of the model if the situation is as assumed but do not allow to check this presupposition. For instance, if the two filter regions in a simple mosaic stimulus are swapped, this usually destroys the transparency impression, but the estimates from the statistical procedures described in the Estimation procedures section would remain unchanged. An additional statistical measure that would allow to detect such cases is the covariance (or correlation) of responses of receptors on different sides of the border between filter and background regions. If \( (A_i^b, P_i^b) \) denote pairs of cone excitations outside and inside the optical filter at the same position along a putative filter contour, then the criterion \( \text{corr}(A_i^b, P_i^b) > t \), where \( 0 \ll t < 1 \) is a
threshold value, may be used to decide whether the contour is actually the border of a transparent overlay or not.

**Comparison of submodels of the filter model**

As discussed in the Filter transparency in simple stimuli section, the proposed filter model is a generic one that may be specialized by choosing a certain method to estimate the illumination or by ignoring some terms of the model equations. There are two possible interpretations of this flexibility: One may either assume that there exists among them a “true model” that best approximates fixed criteria used by the visual system to detect transparent objects or, alternatively, that the submodels describe different strategies of the visual system that may be selected depending on the context, for instance, based on the kind of information available in the proximal stimulus (D’Zmura, Rinner, & Gegenfurtner, 2000).

In order to decide between these alternatives, one needs criteria to evaluate the validity of different submodels. In direct approaches, these criteria would be purely phenomenological, that is, one would prefer the model that best predicts perceived transparency. In the present approach, however, an additional set of computational criteria related to the task of the visual system to detect optical filters may also be used. According to these criteria, a model is considered better if it allows a more accurate or more robust estimation of subjectively relevant properties of optical filters.

If both sets of criteria favor the same model, then this would support the “single best model” interpretation. In a way, this could be considered the simplest case, because it would allow to focus on this specific submodel, whereas otherwise it would not only be necessary to investigate the properties of several submodels but also to identify the triggering conditions that govern model selection.

In the present case, however, the available evidence suggested a more ambiguous situation with respect to the “full” and “restricted” models defined in the Filter transparency in simple stimuli section: Informal observations indicated that the stimuli conforming to the “restricted” model lead to a slightly more convincing transparency impression (this is why we focused on this model in our previous investigation), whereas the “full” model by definition should provide a better description of the color changes caused by optical filters. In the following, we describe a computer simulation conducted to quantify the computational advantages of the “full” model and the results of a formal experiment, which supports the informal observation of an advantage of the “restricted” model. We will also discuss the consequences of these findings in the present approach.

**Accuracy in describing color changes caused by optical filters**

To quantify the relative accuracy of the “full” and “restricted” models in describing color changes in background colors caused by optical filters, we simulated optical filters and determined the error of a fit of both models to these stimuli.

The stimuli had 10 background regions and the filtered colors were computed by using the image generation model described by Equation 1, with CIE standard illuminant D65, filter thickness $x = 1$, refractive index $n = 1.5$, and random choices of absorption and reflectance spectra (frequency limited with $\omega = 1/150$ cycles/nm and $\omega = 1/75$ cycles/nm, respectively; see Appendix A). The absorption spectrum was normalized to the range [0.1, 0.9] to obtain saturated filters, and the reflectance spectra were scaled with random numbers between 0 and 1 to increase the variance in albedo. All spectra were defined in the wavelength range from 400 to 700 nm in steps of 5 nm. The relative errors in region $j = 1 \ldots 10$ of each stimulus are computed as $e_i = |P_i - P_j| / P_j$, where $P_i$ and $P_j$ denote the $i$th color coordinate in filter region $j$ resulting from the simulation and from using the model with the estimated parameters, respectively.

Figure 4 compares the cumulative distribution of the relative error of least square fits of both models to the same colors in 300 simulated filters. These results confirm that the full model describes the color changes caused by optical filters clearly more accurately.

**Experiment 1: Accuracy in predicting perceived transparency**

The aim of Experiment 1 was to verify the informal observation that the reduced model predicts perceived transparency slightly better than the full model. The general logic employed was similar to that used by Nascimento and Foster (2000): The task of the subjects was to compare the goodness of perceived transparency in a stimulus computed using the physical model with a duplicate that was slightly adjusted to conform exactly to either the “reduced” or the “full” filter model.

To compute the stimulus pairs, we first simulated a large number of optical filter in front of a background with 10 different colors ($S_{\text{sim}}$). The procedure used was essentially the same as in the simulation described in the previous section, with the exception that the thickness $x$ and the refractive index $n$ of the optical filter were no longer fixed but chosen randomly from the intervals [0.8, 1.8] and [1, 2], respectively. We then applied the procedure described in the Estimation procedures section [Case 3, with $I = \text{mean (background)}$ and $f = f_M$] to estimate the parameters of the full and reduced filter models. These parameters were then used to calculate
adjusted filtered colors in two similar stimuli $S_F$, $S_R$, conforming exactly to the full and reduced models, respectively. To avoid a large number of indistinguishable stimulus pairs, we selected 150 cases using the criterion that $\max(|X_i - X_j|/X_i) \geq 0.25$, where $X_i$ and $X_j$ are corresponding coordinates of the filtered colors in $S_{sim}$ and $S_R$, respectively. The pairs $(S_{sim}, S_R)$ and $(S_{sim}, S_F)$ were presented in separate trials. Thus, the subjects had to judge a total of 300 pairs.

The stimuli were presented stereoscopically one above the other on a calibrated CRT monitor (ViewSonic P227f, 21”, 1280 x 980 pixels, 75-Hz refresh rate, controlled by a graphics card NVIDIA 8600GT with a bit depth of 8 bits). The stereoscopic viewing was chosen to support a holistic viewing mode and to enhance the depth stratification that usually accompanies transparency impressions. Figure 5 shows a screen shot of the central part of the display. $S_{sim}$ was displayed in the top or bottom row with equal probability. The homogeneous background was set to the mean of the background colors in the stimuli. The square background of the stimuli had a side length of 8.7 cm (6.2°) and the diameter of the circular filter region was 5 cm (3.6°). The horizontal center-to-center distance of the background rectangles was 10.5 cm (7.5°) and that of the circular filter regions was 9.5 cm (6.8°). The subjects viewed the stereo pairs from a distance of 80 cm through a mirror stereoscope (SA200 Screenscope Pro).

For each stimulus pair, the subjects had to choose between three alternatives: (1) both stimuli are indistinguishable, (2) the transparency impression is better in the top row, or (3) the transparency impression is better in the bottom row. With respect to the goodness of the transparency impression, the subjects were instructed to base their judgment on two closely related aspects, namely, how homogeneous the perceived transparent layer appears and how well the relations between the background colors in plain view are retained in the covered region.

**Results**

Figure 6 shows the individual and mean results of three subjects, including one of the authors (FF). The general pattern of results is essentially the same for all subjects. The main difference is in the proportion of stimuli judged
as indistinguishable. In the following, we refer to the mean data, but all statements including those on statistical significance also hold for the individual data. There are three interesting aspects of the results: First, the proportion of “equal” responses is significantly higher in the “full” than in the “reduced” condition ($\chi^2 = 70.58, p < 0.01$). Second, if the two stimuli were perceived as different, then the model-conforming stimuli were clearly preferred over simulated stimuli. According to a binomial test, this preference was significantly different from chance ($p < 0.01$), both in the “reduced” and “full” conditions. Third, the preference for model-conforming stimuli was much higher in the “reduced” than in the “full” condition. A $\chi^2$-test indicates that this preference was also significantly different from chance ($\chi^2 = 7.16, p = 0.007$).

**Discussion**

In the simulation study reported above, we found that the full model describes simulated stimuli better than the reduced model. Thus, the first mentioned experimental result that the proportion of “equal” responses is larger in the “full” than in the “reduced” condition is to be expected if this difference between the two models is large enough to be detected visually. The second result that the model-conforming stimuli usually elicited a more convincing transparency impression than the simulated stimuli supports the assumption that the models correctly describe those aspects of the color changes caused by optical filters that are used by the visual system to infer transparency.

The most interesting result—both with respect to the question motivating the experiment and the implications following from it—is that the preference for model-conforming stimuli over simulated stimuli was more pronounced under the “reduced” condition. This result contradicts the common assumption that a stimulus should appear more realistic the closer it approximates “physical reality”: Stimuli conforming to the “full” model appear less convincingly transparent as those conforming to the “reduced” model, although the “full” model describes color regularities in physical filters clearly better (as is evident from the results reported above and also results reported below in The relation of model parameters to the physical model section).

Although this result does not speak against the idea underlying the filter approach, it is nevertheless surprising, because there seems to be no obvious advantage in...
preferring the “reduced” over the “full” model as both require the same number of parameters and are of similar complexity.

Possible interpretations of the findings

Within the present approach, these results can be interpreted in at least three different ways, namely, (a) that the “reduced” model is actually used, (b) that the “full” and “reduced” models are used simultaneously, or (c) that one of the two models is selected depending on the situation.

The first alternative (a) would be compatible with the “single best” model alternative mentioned above. A possible justification why the visual system may use the “reduced” model as the “best one” despite its slightly inferior performance with respect to the detection task could be that it has implementation-related benefits, for example, that it better fits to existing computational modules that are used in other tasks, or that the model has advantages with respect to the robustness of internal computations.

A reasonable justification of case (b) would be to assume that the “full model” is actually used in some internal computations but that the “reduced” model determines the (maybe less critical) output channel that leads to phenomenal impressions.

A hint at a possible explanation along the lines of option (c) may be gained from the fact that the difference between both models concerns the term related to “direct” reflections of first order (which is missing in the reduced model). The filter model describes only clear transparent objects without scattering, which implies sharp contours both in the mirror image of the environment reflected from the filter surface and in the transmitted background image. A model that allows scattering would instead predict blurred contours in the transmitted and mirror-reflected images. One may argue that the simple stimuli used in the experiment are ambiguous with respect to the amount of scattering: The sharp contours of the background regions indicate a situation without scattering, whereas the uniform direct reflection is rather untypical for this case and may, therefore, hint on the presence of scattering. Thus, a possible explanation may be that the results are biased in the direction of a model with scattering and that the “reduced” model is more similar to that case. This interpretation is also supported by the observation (demonstrated in Figure 2) that a filter seen under diffuse uniform illumination—closely resembling the simple stimuli used in the experiments—does not appear glossy as in cases with nonuniform illumination but hazy like a translucent medium. If this explanation is correct, then the preference for the reduced model may vanish if more complex stimuli with nonuniform direct reflection are used.

The available evidence does not allow to decide conclusively between these alternatives, and as a practical consequence, we usually compare the properties of both models in the remainder of the paper. To gain more insight into these questions, additional empirical tests are necessary. Given the many indications that the full model is superior with respect to computational criteria, it would be especially interesting to check whether case (a) is actually correct. To this end, a useful strategy would be to search explicitly for tasks or situations in which the “full” model allows better predictions. More complex situations with inhomogeneous illumination, where the filter properties and direct reflection can be more easily discerned, may prove to be revealing in this regard.

Remapping the model parameters to a phenomenological space

The derivation of the filter model suggests relations between the parameters of the psychophysical model and properties of the optical filter: \( \tau \) corresponds to the squared transmittance of the filter, whereas \( \delta \) correlates with the amount of direct reflection of incoming light at air–filter interfaces, which in the physical model is described by Fresnel’s equations.

This representation of the model parameters is, however, not very intuitive if one wants to describe their perceptual effects. Therefore, it would be desirable to remap the parameters in a way that captures the relevant perceptual dimensions more properly.

Hue, saturation, and transmittance of a filter

From a phenomenological perspective, the parameter \( \tau \) represents the hue, saturation, and overall degree of transmittance of the filter. A filter with \( \tau = (1, 0, 0) \), for example, could be called a “saturated red” filter in the sense that it transmits predominantly “long-wavelength light” and that a white surface seen through this filter would appear in a saturated red color. A filter with \( \tau = (1, 0.8, 0.8) \) could analogously be called a “desaturated red” filter because a white surface seen through this filter would appear in a desaturated red. By the same logic, a filter with \( \tau = (1, 1, 1) \) would be called white, but in this case, it is more common (and more appropriate) to say that it is completely clear. These examples illustrate that there is a close relationship between \( \tau \) and color codes attributed to surfaces, but it should also be obvious that \( \tau \), although it describes chromatic attributes of the filter, is itself not a color code.

A proper reparameterization of the \( \tau \) parameter should represent the above-mentioned perceptual dimensions in a more intuitive manner. The structural similarity to color codes, on the one hand, and the parameter restrictions on \( \tau \), on the other hand, suggest to use a transformation of a
normalized RGB color space into a perceptually based space, where \( R, G, B \) are replaced with \( C_L, C_M, C_S \). For this purpose, we decided to use the HSV space proposed by Smith (1978). The transformation from \( \tau \) to HSV (see Appendix B for MATLAB code) maps the transmittance vector \( C \) to \( H, S, V \) values, which all lie in the range \([0, 1]\).

Here, “hue,” “saturation,” and “value” are understood as mental representations of filter properties and must not be confused with descriptions of the perceived color in the filter region, which also depends on other factors such as the background seen through the filter and the prevailing illumination. The single case, where these two concepts virtually coincide, is a filter in front of a white background. Obviously, the concept of hue, saturation, and value of a filter is rather complex, and an important aim of the present work was to investigate whether it is useful at all.

### Hue

The parameter \( H \) represents the “hue of the filter.” A rough physical correlate is the dominant wavelength of the light transmitted by an optical filter. The value \( H = 0 \) is taken to be red. By increasing \( H \), the hues take on the colors of the light spectrum from red, over yellow, green, cyan, blue, magenta, to red again. Thus, \( H = 1 \) denotes the same hue as \( H = 0 \). Just as in most color spaces, the hue value has no metrical meaning, that is, the numerical distance between two hue values says nothing about the perceptual difference of these hues. Figure 7 shows a simulated filter, which changes hue in equal steps of \( H \).

“saturated red filter,” thus, means a filter with high selectivity for long-wavelength light. Conceptually, this meaning must be distinguished from the usual use of the term saturation in a color space, where \( S = 0 \) indicates a gray color and where increasing the saturation value means to decrease the “gray content” of the color.

If \( H \) and \( V \) are kept constant (and \( V \) is nonzero), then the \( S \) dimension is at least an ordinal scale, that is, perceived filter saturation increases monotonically with \( S \). Preliminary observations suggest that this scale may be even approximately linear with \( S \). Comparisons of \( S \) values across different hues are more problematic. This problem is discussed in more detail in the Restrictions on layer saturation section and Iso-saturation curves section. Figure 8 shows a simulated red filter that changes its saturation in equal \( S \) steps.

### Transmittance

The value parameter describes the overall transmittance of the filter, with the poles \( V = 0 \) (“no transmittance”) and \( V = 1 \) (“full transmittance”). In the HSV model, \( V = \max(\tau_L, \tau_M, \tau_S) \). If the additive component due to direct reflection is zero (i.e., \( \delta = 0 \)), then a filter with \( V = 1 \) is actually invisible if the filter is also achromatic (\( S = 0 \)), and the color of filters with \( S > 0 \) is of high purity. If \( V \) decreases below 1, then the filter appears increasingly darker and in the limit completely black. Figure 9 shows a simulated achromatic filter with varying transmittance value \( V \).

### Saturation

The parameter \( S \) describes the “saturation” of the filter. Its meaning can best be understood with reference to the physical correlate: A saturation of zero means a completely achromatic filter, which transmits light of all wavelength equally, and increasing saturation means increasing wavelength selectivity of transmittance. A "saturated red filter," thus, means a filter with high selectivity for long-wavelength light. Conceptually, this meaning must be distinguished from the usual use of the term saturation in a color space, where \( S = 0 \) indicates a gray color and where increasing the saturation value means to decrease the “gray content” of the color.

If \( H \) and \( V \) are kept constant (and \( V \) is nonzero), then the \( S \) dimension is at least an ordinal scale, that is, perceived filter saturation increases monotonically with \( S \). Preliminary observations suggest that this scale may be even approximately linear with \( S \). Comparisons of \( S \) values across different hues are more problematic. This problem is discussed in more detail in the Restrictions on layer saturation section and Iso-saturation curves section. Figure 8 shows a simulated red filter that changes its saturation in equal \( S \) steps.

### Filter clarity

It appears less intuitive how the “direct reflection” parameter \( \delta \) influences the perceived properties of the filter. This is not without reason, because the perceptual effect depends both on \( \delta \) and the nature of the illumination \( I \).
In the present context, we restrict ourselves to the special case of diffuse uniform illuminations. Under natural viewing conditions, this would be approximately correct if the filter is indirectly illuminated by a clear sky or by a uniformly colored room. In this case, the light that is directly reflected from the filter’s top surface is uniform and adds to the spatially structured light reflected from the background. With increasing $\delta$, the filter looks less clear but instead appears increasingly hazy (see Figure 10). This motivates to use the term “filter clarity” to describe the corresponding impressions. With nonuniform illuminations, the direct reflection is, in general, a distorted image of the surround, and the filter surface appears more or less glossy (thus, with respect to this case, it would be more appropriate to use the term “filter glossiness”). These perceptual effects can be appreciated in Figure 2.

The “direct reflection component” is due to Fresnel reflection at interfaces between media with different refractive index. The direct reflection of first order is the light that is reflected at the top surface of the filter, that of second order is the light that is reflected once at the lower inner surface of the filter and then leaves the filter at the top surface, and so on. The direct reflection components are, in general, spatially and (with the exception of first-order reflection) chromatically distorted images of the environment and add to the light transmitted through the filter from the background. In optical filters, the amount of direct reflection increases with the angle between the direction of the incoming light and the surface normal and with the size of the refractive index. The strength of reflections of order 2 and higher also depends on the transmittance of the filter: The higher the order, the smaller the strength, because the length of the path the light travels inside the filter until it leaves the top surface increases with order, and the Bouguer–Beer law states that transmittance along a path inside the filter decreases exponentially with path length.

With respect to the validity of the specific form of the clarity parameter $C$ proposed here, it is interesting to note that it is closely related to a measure found by Singh and Anderson (2002) to determine the perceived properties of the transparent layer in achromatic stimuli. They
simulated two transparent layers in front of identical sinusoidal gratings by changing the mean and contrast of the sinusoidal grating inside a central “filter region.” In the standard stimulus, the mean and contrast of the “filter” were fixed. The mean of the comparison “filter” was systematically varied, and for each specific value, the task of the subjects was to match the perceived transmittance of standard and comparison filters by adjusting the contrast of the latter. The results revealed that the subjects always chose a setting that made the ratio of the Michelson contrast in the filter region to the Michelson contrast in the background equal in both stimuli. This finding is rather difficult to explain in the framework of the episcotister model to which the authors refer and was even presented by them as an argument against physically inspired models. In the present framework, however, it can easily be understood by assuming that their subjects matched the perceived clarity of the filter, because the clarity parameter is closely related to the ratio of the Michelson contrast in the filtered region to the Michelson contrast in the background. This can be seen if the formula for \( C \) is inserted in Equation 10 to yield

\[
C = \frac{1}{(1 + \delta)} = \frac{(A_i + B_i)(P_i - Q_i)}{[(A_i - B_i)(P_i + Q_i)]}
\]

That the subjects actually matched the clarity of the perceived transparent layer instead of its “transmittance,” as requested, is also suggested by the fact that the means of the two “filters” (that determine their perceived darkness) were different.

The relation of model parameters to the physical model

Successful vision-guided interaction with the environment requires that the mental models underlying the interpretation of the visual input capture important regularities of the external world, because this is a necessary prerequisite to construct mental representations of objects and their properties that are stable under varying viewing conditions and contexts.

In Faul and Ekroll (2002), we found in a simulation study that the filter model given in Equations 6 and 7 can be well fitted to the colors of simple four-color stimuli that were generated using the physical filter model described by Equation 1. This result indicates that the model captures essential structural relationships between background and filtered colors typically found in “filter situations” and that the degree of consistency with the model equations can serve as a useful criterion to decide whether a given stimulus should be classified as transparent or not.

If the model actually provides a useful description of transparency situations, one would further expect that the model parameters change in a regular and predictable way with variations in important physical properties of the related distal object and that the parameters that are estimated for a fixed distal object remain approximately constant across typical variations of context, viewing conditions, and changes in illumination. In the following, we report results of computer simulations that were conducted to investigate to what extent these attributes apply to the filter model.

Model parameter and filter thickness

Changes in filter thickness (along the viewing direction), either due to the specific form of an object (e.g., along a wedge) or due to a different perspective on the object (e.g., top or side view of a flat disk), are quite common under natural viewing conditions. Thus, an interesting question is how the model parameters change with the thickness of the filter.

To investigate this, we used the physical model given in Equation 1 to calculate large samples of stimuli with...
$N = 10$ background/filter color pairs $(A', P')$ and then applied the global estimation procedure described as Case 3 in the \textit{Estimation procedures} section to calculate the model parameters for each simulated stimulus. The fixed scene properties in a sample were the absorption spectrum of the filter (a randomly generated frequency-limited spectrum with limiting frequency $\omega = 1/150$ cycles/nm), the illumination spectrum (CIE daylight spectrum D65), and the refractive index $n$, which was set to a fixed value of $\geq 1$. To avoid a bias in favor of low saturation filters (one

![Figure 11](http://jov.arvojournals.org/)  

\textbf{Figure 11.} Stereo pairs demonstrating the color changes in the filter region due to a linear change in filter thickness from 0.01 at the top to 3 at the bottom. The physical model was applied to simulate an optical filter with a fixed absorption spectrum and a refractive index of 1.2. CIE illuminant D65 and frequency-limited reflection spectra with limiting frequency $\omega = 1/150$ cycles/nm were used. The left image pair is for uncrossed fusion, and the right image pair is for crossed fusion.

![Figure 12](http://jov.arvojournals.org/)  

\textbf{Figure 12.} Typical examples of the effects of varying filter thickness on the parameter of the filter model and the transmittance spectrum of the optical filter. In the top row, a refractive index of 1.0 was used; in the two lower rows, it was 1.5. The middle and bottom rows show the results for the same filter using the full and reduced models, respectively. The scatter plots in the first four columns show the distribution of 200 parameter values that were calculated using an optical filter with the absorption spectrum shown as black curve in the rightmost panel, random choices of reflection spectra for 10 background regions, and random choices for the filter thickness between 0.1 and 3. The colored lines in the rightmost panels show the squared transmittance spectrum for the absorption spectrum shown in black and three thicknesses (0.1 = red, 1 = green, 2 = blue, 3 = cyan). See text for more details.
may also say: to induce a bias to high saturation filters), the absorption spectrum was shifted and scaled to set its minimum $\alpha_{\text{min}}$ and range $\alpha_{\text{range}}$ to specific values (0.1 and 0.8, respectively, if not otherwise specified). The scene properties that were chosen differently for each stimulus of the sample were the reflection spectra of the $N$ background regions (randomly generated frequency-limited spectra with $\omega = 1/75 \text{ cycles/mm}$) and the filter thickness, which was randomly chosen between 0.1 and 3. To increase the luminance range of the corresponding surface colors, the reflectance spectra were multiplied with a uniformly distributed random number between 0 and 1.

The stereo pairs in Figure 11 demonstrate how the colors in the filter region change with increasing filter thickness. Figure 12 displays typical simulation results in a sample. The results for samples with refractive index 1 (no direct reflection) are shown in the top row and those for samples with refractive index 1.5 (approximately that of glass) are shown in the two bottom rows. The middle and bottom rows compare the results of the full and reduced model, respectively. In each row, the first four panels show the effect of changes in filter thickness on the four parameters of the filter model, and the rightmost panel illustrates the effect in the image generation model. The black curve in the rightmost panels is the absorption spectrum of the filter material and the colored curves are the corresponding squared transmittance spectra for four different thicknesses. It can be seen that with increasing thickness, the maximum of the transmittance spectrum decreases and the wavelength selectivity gets more pronounced. The latter effect of “spectral sharpening” is due to the fact that transmittance decreases exponentially with thickness as stated by the Bouguer–Beer law.

### Hue

In our simulations, the estimated hue of a filter with fixed absorption spectrum usually changed only slightly with varying thickness, as long as the saturation value was not very low. The examples shown in Figure 12 are quite representative in this respect. To quantify the size of the effect, we calculated the mean standard deviation of hue values in a large number of cases like the ones depicted in the leftmost column of Figure 12: For each of six different conditions (two refractive indices combined with three minimum/range conditions for the absorption spectrum), 200 samples of 200 stimuli each were generated using the methods described above. Then, in each sample, the SD of the estimated hues was calculated. To avoid problems with the circular nature of hue values ($H = 0$ is identical to $H = 1$), we first determined the mean hue from the angle of the centroid of the distribution of $HS$ pairs in a polar plot in the $H/S$ plane and then determined the SD of estimated hues remapped to the interval $[-0.5, 0.5]$ around this mean hue. The results in Table 1 show that the mean SD of estimated hues for saturation values $S > 0.2$ was less than 0.02 with the full model. Using the reduced model, the SDs were even smaller. Given that $H$ values determine the perceived hue of the filter, these results suggest that the effect of changes in the filter thickness on perceived hue is, in most cases, negligible.

#### Transmittance

As can be seen in the third column of Figure 12, the transmittance parameter $V$ decreases exponentially with filter thickness. This holds for both the full and reduced models. More specifically, we found that for a given filter the function $V(x) = a \exp(-bx) + c$, with $a, b > 0$, described the dependence of the transmittance parameter on filter thickness $x$ almost perfectly. We performed least square fits of this model to 200 samples of 200 stimuli, each under the same six conditions described above. The

### Table 1: Mean and maximum standard deviation of hue estimates using the full model in 200 samples of 200 stimuli, each under six conditions: $n$ denotes the refractive index; $\alpha_{\text{min}}$ and $\alpha_{\text{range}}$ denote the minimum and range of the absorption spectrum. The columns under $S > 0.2$ show the mean and maximum of SD for stimuli in which the saturation was greater than 0.2; the other two columns show the same values for all estimates.

<table>
<thead>
<tr>
<th>Condition</th>
<th>SD of hue values</th>
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<tbody>
<tr>
<td></td>
<td>$S &gt; 0.2$</td>
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<tr>
<td>$n$</td>
<td>$\alpha_{\text{min}}$</td>
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<td>1.0</td>
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### Table 2: Means and SD of estimated parameters of $V(x) = a \exp(-bx) + c$, which describes the dependence of filter model parameter $V$ on thickness, in 200 samples of 200 stimuli each, under six conditions: $n$ denotes the refractive index; $\alpha_{\text{min}}$ and $\alpha_{\text{range}}$ denote the minimum and range of the absorption spectrum.

<table>
<thead>
<tr>
<th>Condition</th>
<th>Mean estimate ± SD</th>
</tr>
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<tbody>
<tr>
<td>$n$</td>
<td>$\alpha_{\text{min}}$</td>
</tr>
<tr>
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restricted model

200 samples of 200 stimuli each, under six conditions: $n$ denotes the refractive index; $\alpha_{\text{min}}$ and $\alpha_{\text{range}}$ denote the minimum and range of the absorption spectrum.
The exponential $\exp(-x)$ tends to 1 if the thickness of the optical filter. The mean $R^2$ was always larger than 0.996 indicating rather good fits. The parameter $v$, which controls how fast the saturation increases with thickness, has a maximum at the largest range of the absorption spectrum; it is clearly lower for the two narrower ranges, irrespective of whether the range is at the low end (highly transmissive filters) or the high end (dark filters). For conditions with refractive index $n = 1$, the mean $R^2$ values were always larger than 0.996 indicating rather good fits.

We found that the increase of saturation with thickness $x$ can often be well described by the equation $S(x) := u[1 - \exp(-vx)]$, with $v > 0$ and $0 < u \leq 1$. The latter condition is motivated by the fact that $S$ is restricted to the interval $[0, 1]$. To test the appropriateness of this description, we again performed least square fits to 200 samples of 200 stimuli each, under the six conditions described above. The mean $R^2$ of the fit and the mean and $SD$ of the parameter estimates are summarized in Table 3. Under conditions with refractive index $n = 1$, the mean $R^2$ values were always larger than 0.996 indicating rather good fits. The parameter $v$, which controls how fast the saturation increases with thickness, has a maximum at the largest range of the absorption spectrum; it is clearly lower for the two narrower ranges, irrespective of whether the range is at the low end (highly transmissive filters) or the high end (dark filters). For conditions with refractive index $n = 1.5$, essentially the same pattern is found for the full model: The distributions of the parameter estimates are almost identical and the goodness of fit is only slightly reduced. This is clearly not the case in the reduced model. On average, the fits of the reduced model are clearly worse (lower $R^2$ values) and lead to lower overall saturation values (low values of parameter $u$) and faster increases of the saturation with thickness (high values of parameter $v$). With increasing thickness, the relative contribution of direct reflection to the total light emanating from the filter region increases, because the amount of direct reflection is almost independent from thickness. Thus, all colors in the filter region are shifted in the direction of the illumination color. In the reduced model, this effect is "misrepresented" as a change in the filter properties, partly as a decrease in filter saturation and partly as a decrease in filter clarity. Especially for low overall saturation, this may even result in a saturation distribution with a single peak at mean thickness, where both lower and larger thicknesses lead to decreasing saturation.

### Table 3. Means and SD of estimated parameters of $S(x) = u[1 - \exp(-vx)]$, which describes the dependence of filter model parameter $S$ on thickness, in 200 samples of 200 stimuli each, under six conditions: $n$ denotes the refractive index; $a_{\text{min}}$ and $a_{\text{range}}$ denote the minimum and range of the absorption spectrum.

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<thead>
<tr>
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<th>$v$</th>
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<td>0.87 ± 0.21</td>
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</tbody>
</table>

**Saturation**

Saturation normally increases with filter thickness. Typical examples can be seen in the second column of Figure 12. Very thin filters show little wavelength selectivity irrespective of the form of the absorption spectrum $a(\lambda)$. This is a direct consequence of the fact that the exponential $\exp(-a(\lambda)x)$ in the Bouguer–Beer law tends to 1 if the thickness $x$ tends to zero. With increasing thickness, the filter transmits less light [unless $a(\lambda) = 0$] and is—due to “spectral sharpening”—increasingly wavelength selective [unless $a(\lambda) = \text{const}$] and, thus, looks increasingly darker and more saturated. For refractive indices $n > 1$, there is also an additive component due to reflection of the illumination at filter–air interfaces. The dominant contribution in this component is the reflection at the top surface of the filter, and this part does not depend on filter thickness. If the illumination is approximately neutral, then this component has an additional desaturating effect.

---

**Clarity**

The clarity parameter of the filter presumably reflects the amount of direct reflection. In simulated filters with a refractive index $n = 1$, which implies zero direct reflection, we always found estimates of the clarity parameter near the maximum value of 1 (see top row in Figure 12). This is exactly what one would expect in this case. For refractive indices $n > 1$, the estimated clarity values should be less than 1 and approximately constant with changes in thickness, because the first-order reflection at the top surface, which constitutes the dominant part of the total direct reflection, does not depend on filter thickness. The typical results for $n = 1.5$ obtained with the full model conform closely to this expectation (see middle row in Figure 12). This is not true for the results obtained with the reduced model shown in the bottom row of Figure 12. Here, the clarity values decrease with thickness. 

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respect to saturation, we find that the spatial distributions of the colors in the background (green) and filter (red) regions. The window sizes used in the simulation were \( h = 35, w_o = w_i = 3 \) pixels.

This again indicates that the parameters of the full model reflect properties of the physical parameters more accurately than the reduced model.

**Retrieving information on thickness from transmittance and saturation**

The close relationship of the transmittance and saturation parameters to filter thickness suggests that it should be possible to retrieve information on filter thickness from the spatial distribution of \( V \) and \( S \).

With respect to transmittance, our results imply that \( -\log(V) = bx - \log(a) \) is a linear function of filter thickness \( x \), if the small contribution of parameter \( c \) is ignored. With respect to saturation, we find that \( -\log(u - S) = vx + \log(u) \) is a linear function of \( x \). A problematic point in this case is that \( u \) is normally not known. However, the range of \( S \) is usually relatively small, and thus, a potential solution may be to either use \( u = 1 \) or to take the maximum saturation in a spatial region as an estimate for \( u \). These results suggest that the spatial distributions of \( -\log(V) \) and \( -\log(u - S) \) provide an independent basis for an estimate of the changes in filter thickness and thus, indirectly, of filter form. This information may, for instance, be exploited in spatial correlation algorithms (which are invariant to linear transforms of the input) that can be used to detect an optical filter based on form information.

In order to probe the reliability of this information, we simulated optical filters of varying thickness in front of the four different backgrounds shown in Figure 14 and tried to recover the depth from parameter estimates of the filter model. The reflectance spectra of the color in the background images were frequency limited \((\omega = 1/75 \text{ cycles/nm})\) and the illumination was fixed \((\text{CIE D65})\). These “spectral images” were generated from RGB bitmaps using the method described in Appendix A. The physical filter model was then used to compute the colors inside the filter region using the reflection spectra of the image pixels, the constant absorption spectrum of the simulated filter, a fixed refractive index, and thickness values that varied sinusoidally along the filter contour.

The methods outlined in the Estimation procedures section \([\text{Case 3}, \text{with } f = f_M]\) were then used to compute local estimates of transmittance \( V(c) \) and saturation \( S(c) \) at point \( c \) along the contour of the simulated optical filter. The inputs to this algorithm were the mean and standard deviation of background and filtered colors inside a window centered around \( c \) (see Figure 13). To avoid problems at the image borders, the images were first padded with mirror reflections of the image. Finally, to compensate for noise in the local parameter estimates, they were smoothed by applying a box filter, i.e., by computing the moving average of the \( V \) and \( S \) estimates along the contour. The filter length was 81 pixels and the unpadded length of the filter contour was 512 pixels.

**Results**

Figure 14 shows some typical results obtained with the full model using a refractive index \( n = 1.3 \) and a sinusoidal thickness function with 3/2 cycles along the contour. In these examples, the correlation of \( -\log(V) \) and \( -\log(1 - S) \) with the true thickness is rather high, throughout. We performed more extensive tests with different refractive indices \((n = 1 \text{ and } n = 1.5)\), different filter absorption spectra \((\text{frequency limited with } \omega = 1/150 \text{ cycles/nm})\), and different thickness functions \((\text{varying in mean } = 1, 2, 4, \text{ frequency } = 1/2, 1, 2 \text{ cycles, and amplitude } = 0.25, 0.5, 0.75 \text{ of mean})\). The main results were:

1. The estimation based on saturation works almost perfectly \((r > 0.9)\) for \( n = 1 \) but is much less reliable for \( n = 1.5 \), especially if the reduced model is assumed. In the latter case, we often found strong negative correlations between estimated and true thicknesses.
2. The estimation based on transmittance proved more robust. In most cases, the estimate was rather good. The only exceptions were filters with both low spatial frequency and low mean absorption.
3. The most reliable estimates were found with the vertical stripes image shown at the top right position in Figure 14.

A first explanation for the advantage of the striped image is that it optimally fulfills the implicit assumption made in the estimation algorithm that the background seen through the filter is identical to that seen in plain view. A second reason is that the estimates are based on a relatively large sample of different colors, due to the dense pattern along the contour. These considerations also motivate a long, narrow window along the filter contour: The narrow width helps fulfill the “same background” assumption, and the long extension along the border is in favor of a large sample with many different colors.
Model parameters and filter refraction

An important parameter of the physical model of optical filters is the refractive index. In the general case, it determines the amount of direct reflection and the geometrical distortions of the background seen through the filter. It is, therefore, interesting to know how changes in the refractive index influence perceived transparency and the parameters of the filter model.

The stereo pairs in Figure 15 show the image of an optical filter simulated in the reduced situation assumed in the derivation of the filter model (see the Image generation section). All parameters of the filter are constant, with the exception of the refractive index, which changes linearly from 1 at the top to 1.7 at the bottom. Such gradual changes of the refractive index do not normally occur in natural objects; the simulation should, therefore, be understood as a means to visualize changes in the filtered colors that accompany changes in the refractive index. This visualization suggests that in the simple situation assumed in the simulation, the main perceptual effect of increasing the refractive index is that the filter appears increasingly less saturated and less clear.

To investigate how changes in the refractive index of the optical filter influence the four parameters of the filter model, we employed essentially the same method that was used to investigate the influence of filter thickness. That is, we again simulated filter situations with $N = 10$ background colors where all properties were fixed, except for the background colors and the parameter of interest (here the refractive index), which were randomly chosen. The global estimation procedure described as Case 3 in the Estimation procedures section was then used to estimate for each simulated stimulus the parameters of the filter.
model. All conditions were identical to those described in the Model parameter and filter thickness section, except for the choice of saturation instead of thickness as the randomly varied variable.

**Figure 16** displays typical simulation results. In the top row, the parameter estimates of the full model are plotted; in the bottom rows, those for the reduced model are plotted. In each row, the first four panels show the effect of changes in the refractive index on the four parameters of the filter model, and the rightmost panel shows the absorption and the squared transmittance spectrum of the filter material used in this particular simulation.

**Figure 17** gives a more complete picture of the general results of our simulation. It shows the difference in the mean parameter estimates for \( n = 1 \) and \( n = 1.7 \) in 50 samples of 200 stimuli each. In each subplot, a different thickness of the filter was used. This difference between estimates at \( n = 1.0 \) and \( n = 1.7 \) under otherwise identical conditions is a good indicator of the strength of the effect, because (as is illustrated in **Figure 16**) all curves either are approximately constant or show a monotonic decline with increasing refractive index.

Our results suggest that changing the refractive index has the largest influence on the clarity parameter. The clarity parameter assumes values near the maximum of 1.
The decrease observed with the reduced model depends also on the overall transmittance and is especially strong for dark filters with low transmittance. The mean saturation decreases with an increasing refractive index, whereas it remains almost constant when the full model is used. The decrease observed with the reduced model is further increased (see the fourth panel from the left in Figure 16) when the refractive index is increased. This decrease is found with refractive indices lower than approximately 1.1 and then decreases in an almost linear fashion when the refractive index is further increased (see Faul & Ekroll, 2002, p. 1076). Here, we consider a second kind of restriction on \( \tau \) that results from the fact that the model describes relationships between color codes in the input. To gain insight into the nature of this restriction, it is helpful to consider the reduced model \( P_i = \tau(A_i + \delta I) \), which can be written as \( P_i = \tau X_i \), where \( X := A + \delta I \) is a weighted sum of the color codes of background \( A \) and illumination \( I \) and thus itself a valid color inside the color cone. \( P \) is a stimulus color and thus by necessity also a valid color. This implies that \( \tau \) is restricted to values that map \( X \) to colors inside the color cone. The parameter values \( \tau = (x, x, x) \), \( x \geq 0 \)—corresponding to neutral filters—are always possible, because they map \( X \) to the same vector in color space, whereas \( \tau = (0, 1, 1) \) is an example of an impossible parameter value, because this would result in a zero \( L \) and a nonzero \( M \) cone excitation in \( P \), which is not realizable due to correlations between the \( L \) and \( M \) channels. In this way, the parameter vector \( \tau \) “inherits” restrictions on color codes.

To further explore this second kind of restriction, it is more intuitive to refer to the reparameterization of \( \tau \) in terms of “hue” \( H \), “saturation” \( S \), and “transmittance” \( V \) proposed above. In this parameter space, the “physical restrictions” on \( \tau \) translate into the restrictions \( 0 \leq H, S, V \leq 1 \) and the “color space restrictions” into limits for the maximum value for \( S \). The maximum \( S \) varies with \( H \) and can best be visualized in a polar plot of \( H \) and \( S \), where \( H \) is mapped to the angle and \( S \) to the radius. In such plots, the set of maximal \( S \) encloses a subregion of the unit disk (see Figure 18). Only points inside this region correspond to valid \((H, S)\) pairs, which combined with admissible transmittance values \( V \in [0, 1] \) correspond to valid \( \tau \). Points on the border of the region correspond to values of \( \tau \) that map \( X \) to filtered colors \( P \) on the boundary surface of the color cone, and the point \( S = 0 \), which is always contained in the region, corresponds to neutral filters.

The form of the region depends on the hue and saturation of the color code \( X \), which in turn depends on the color codes of background and illumination. An important special case are achromatic \( X \). In this case, the

Figure 17. Mean difference in the parameters of the full and reduced filter models for different refractive indices. Each data point shows the difference between the mean value of the respective parameter found with \( n = 1.7 \) and \( n = 1.0 \) in 50 samples of 200 stimuli each. The error bars enclose \( \pm 2 \) SD. The continuous lines between the different parameters are inserted to make the pattern more obvious. See text for details.
boundary of the region (shown in gray in Figure 18) is numerically identical to a transformation of the boundary of the color cone to the HS space. To compute the transformed values, the cone sensitivities \( l(\lambda) \), \( m(\lambda) \), and \( s(\lambda) \) are first scaled in such a way that they have equal area and values \( \leq 1 \) and then the normal RGB to HSV transform is applied to the spectral colors. For chromatic \( X \), the degree to which the “valid region” deviates from that of the achromatic case (the “achromatic region”) increases with the saturation of \( X \) (compare the red and blue regions in Figure 18).

In the filter model, we assume that the stimulus contains at least two different background colors. It is clear that admissible \( \tau \) must be simultaneously compatible with all corresponding \( X \). The solution space for \( H \) and \( S \) is, therefore, the intersection of the valid regions for all \( X \). It can be seen in the rightmost panel of Figure 18 that even the combination of only two (saturated) colors may restrict the solution space considerably. The intersection region usually deviates much less from the “achromatic region” than the source regions (in the sense that the nonoverlapping area is smaller) and may even lie completely inside.

Large deviations of the “valid region” from the “achromatic region” only occur for saturated \( X \). Thus, it is interesting to consider factors that may limit the saturation of \( X = A + \delta l \). A first factor is related to the concept of optimal color stimuli (Wyszecki & Stiles, 1982) for a given illumination and leads to limits on the saturation of background colors \( A \). The key observations are (a) that surfaces with a unit spectrum are the brightest possible and that they always reflect light with the chromaticity of the illumination, and (b) that monochromatic reflection spectra are the most saturated ones for a given hue but also the darkest possible. If one determines for each luminance level between these two extremes the range of possible chromaticities (the MacAdams limits, see p. 179 ff. in Wyszecki & Stiles, 1982), one gets the Rösch color solid, which encloses (up to a factor related to the intensity of the illuminant) all possible surface colors under the given illumination. Figure 19 shows the MacAdams limits for CIE standard illuminants A and D65 in the CIE xy chromaticity plane. The restrictions on possible chromaticities are even more severe if we assume frequency-limited reflection spectra that putatively resemble “natural” spectra. Figure 19 shows examples of the color solids for different frequency limits. It illustrates that the restrictions get tighter with decreasing frequency limit. A second factor that limits the saturation of \( X \) is due to the term \( \delta l \): With increasing values of \( \delta \), that is, with increasing amounts of direct reflection, the relative contribution of the illumination color to \( X \) increases and the chromaticity of \( X \) is shifted nearer to that of \( l \). As this term is identical for all background colors, the gamut of possible chromaticities shrinks around the chromaticity of \( l \).

### Distribution of filter parameters

Above, we discussed various causes that may lead to restrictions on possible values for \( \tau \). It is, however, not easy to envision how these different factors combine in typical scenarios. Therefore, we simulated optical filters using realistic choices of the physical parameters to determine the distribution of the parameters of the filter model, in particular the distribution of filter saturation.

### Methods

The general method was to simulate a large number of optical filters with \( N \) background spectra. The procedure described as Case 3 in the Estimation procedures section (with \( f = f_g \)) was then used to compute from the color codes resulting from the simulation the parameters \( \tau \) and clarity of the filter model. The parameter \( \tau \) was then
transformed to $H$, $S$, and $V$ values. In each case, the $N$ reflection spectra of the background were chosen randomly from the set of frequency-limited spectra with $\omega = 1/75$ cycles/nm, and the absorption spectrum of the filter was a randomly chosen frequency-limited spectrum with $\omega = 1/150$ cycles/nm. The remaining parameters, that is, filter thickness, refractive index, $N$, the illumination, and the type of model (full vs. reduced) were varied systematically in separate simulations.

**Results**

We first discuss the results depicted in Figure 20: Each row shows the distribution of the four parameters of the filter model, estimated from 30,000 simulated optical filters. The number of background colors was always $N = 10$.

Row (a) shows the distribution of the parameters of the full filter model. The optical filter was simulated under CIE standard illuminant D65 and had a refractive index of 1.5 and a thickness randomly chosen between 0.5 and 1.5. Under these conditions, $H$ and $S$ lie inside a relatively small, roughly elliptical region around the zero saturation point $S = 0$. The frequency plot in the leftmost panel shows that there is, in this case, a strong bias toward achromatic filters. The estimated value of the clarity parameter is always clearly less than 1, as is to be expected with a refractive index greater than 1, and lies inside a small interval. The clarity estimate is also independent of the estimated transmittance value of the filter.

The conditions in row (b) are identical to those in row (a), with the single exception that the randomly chosen absorption spectrum of the optical filter was always shifted and scaled in such a way that its minimum and maximum values were 0.1 and 0.9, respectively. This introduces a bias toward higher saturated filters. Accordingly, the size of the elliptical region is increased somewhat, but the ellipse, nevertheless, lies well inside the “achromatic region.” The manipulation of the absorption curve also led to a slightly increased transmittance value, whereas the estimates of the clarity values were unaltered.

The conditions in rows (c)–(e) each deviate from those realized in row (b) in a single variable. In row (c), the difference is that the refractive index is now set to 1. As expected, this leads to estimated clarity values close to 1. The mean estimated transmittance is also slightly enhanced, whereas the effect on the distribution of the hue and saturation parameter is negligible. In row (d), the single difference is that the parameter of the reduced instead of the full model was estimated. In this case, the general form of the hue and saturation distribution is retained, but the saturation values are clearly lower than those found with the full model. The estimated clarity values are also much lower and are no longer independent from estimated transmittance. In row (e), the difference is that the CIE standard illuminant A is used instead of standard illuminant D65. Although this manipulation has a noticeable effect on the distribution of hue and saturation, the general form of the distribution is very similar to that shown in row (b). The distribution of the clarity and transmittance parameter is virtually unaffected.

Figure 21 shows the results of simulations with $N = 2$ background colors. The conditions in rows (a) and (b) are otherwise identical to those realized in rows (b) and (c) in Figure 21, respectively. Although the general form of the distributions found with $N = 2$ is similar to those obtained with $N = 10$, there are also marked differences. The most
Figure 20. Each row shows the distribution of hue $H$, saturation $S$, transmittance $V$, and clarity parameter $C$ estimated from the simulated proximal stimulus of optical filters under different conditions. The two middle panels show scatter plots; the two outer panels show contour plots of the frequency distribution. See text for details.
obvious effect is that the precision of the clarity estimate is strongly reduced with $N = 2$ background colors. Another striking effect is that the border of the hue and saturation distribution is fuzzier: There are many estimates with large saturation values well outside the elliptical region. Row (c) shows the results of a simulation in which we tried to maximize the saturation of the estimated filter. The filter thickness was set to 3, the refractive index was set to 1, and the absorption spectrum was scaled and shifted to obtain values in the interval $[0.001, 0.999]$. This introduces a bias toward relatively dark and strongly saturated filters. However, even in this case, the saturation of most filters was within the “achromatic region,” outlined by the red curve.

Discussion

The simulation results depicted in Figures 20 and 21 indicate that under realistic conditions filter saturation is highly restricted. This result is not surprising given the considerations in the Restrictions on layer saturation section, which have revealed that nominally high saturation values are, in principle, possible but that they can only occur under very specific conditions. Extreme saturation values are especially improbable in complex backgrounds with many different colors. In our simulations, such cases are only observed in conditions with $N = 2$ background colors that are shown in Figure 21.

In an $H/S$ polar plot, the hue and saturation values lie inside an elongated elliptical region, where the angle between the major axis and the $x$-axis is roughly 45 deg. The general form and orientation of this elliptical region is rather robust against changes in the refractive index (Figure 20c), the model (Figure 20d), and the illumination (Figure 20e). As will be shown in the Iso-saturation curves section, this characteristic elliptical form is also found in iso-saturation curves.

A comparison of the simulation results obtained with $N = 10$ background colors shown in Figures 20b and 20c with the corresponding ones with $N = 2$ shown in Figures 21a and 21b clearly demonstrates the advantages of including

![Figure 21](http://jov.arvojournals.org/)
more background and filtered colors when estimating the filter parameters. The increase in precision is especially obvious with respect to the clarity parameter.

### Iso-saturation curves

The analysis in the previous section revealed that filter saturation is restricted to a range between zero and a maximum value \( S_m \leq 1 \) that depends on hue. This maximum defines the boundary of a valid region in the \( H/S \) diagram and we have discussed various factors that influence the form of this boundary.

These boundaries are not directly related to perceived saturation but instead only dictate which (nominal) saturation values can be realized in a given situation. In this section, we present the results of an experiment in which we investigated how perceived saturation depends on hue. More specifically, we determined iso-saturation curves in the \( H/S \) diagram. By definition, all filters lying on these iso-saturation curves appear equally saturated irrespective of filter hue.

### Experiment 2

Interestingly, informal observations strongly suggest that the relationship between nominal and perceived saturation is identical for all hues up to a stretching of the nominal scale. Capitalizing on this regularity, we asked subjects to compare the full saturation scales of different hues (see Figure 22) rather than just two single saturations.

We restricted ourselves to achromatic backgrounds. A total of 24 stimuli were arranged in four rows of 6 stimuli each. All stimuli had the same background. It was divided into 10 sectors that were assigned to 10 different gray colors (CIE 1931 \( x = 0.302, y = 0.308 \) equidistant in luminance between 17.4 and 36.8 cd/m\(^2\)). To minimize any influence of background configuration, the assignment of luminance to background area was random in each trial. The standard stimuli in rows 1 and 3 constituted a series of 12 filters with fixed hue value \( H = 0.345 \) (greenish) in which the saturation increased from \( m/12 \) to \( m \) in equidistant steps, where \( m \) is the maximally realizable saturation at the standard hue. The hue of the standard stimuli was the one at which the maximum nominal saturation value realizable on our monitor was minimal. The stimuli in rows 2 and 4 contained a series of comparison filters with a different hue, selected from a set of 67 hue values that ranged from \( H = 0 \) to \( H = 0.99 \) in equidistant steps of 0.015. The subjects’ task was to adjust the comparison filters the endpoint \( u \) of an equidistant saturation scale between \( u/12 \) and \( u \) in such a way that the perceived saturation of the comparison filters was as similar as possible to that in the standard filters over the whole saturation scale. Thus, in an ideal case, the perceived saturation of each comparison stimulus along the scale should be identical to that of the standard stimulus depicted immediately above it.

The remaining filter parameters “transmittance” \( V \) and “clarity” \( C \) were varied in three combinations: \((V = 0.5, C = 1.0), (V = 0.5, C = 1.0), (V = 1.0, C = 0.5)\). These three conditions were applied in separate sessions, and in each session, the settings for all 67 hues were repeated 3 times. This resulted in a total of 201 trials for each subject, which were completed in random order.

The equipment was the same as in Experiment 1. The stimuli were presented on a calibrated CRT monitor at a viewing distance of 80 cm. The screen area that was not covered by the stimuli was set to the mean color of the background colors. The square background of the 24 stimuli had a side length of 6.2 cm (4.4°) and the diameter of the circular filter region was 3.8 cm (2.7°).

Six color-normal subjects (as tested with the Ishihara plates) participated in the experiment. Five of them performed all three conditions, and one additional subject completed only condition \((V = 1.0, C = 1.0)\).

### Results

As expected, the subjects reported that they found it easy and natural to match the whole saturation scale simultaneously. That this was indeed possible is also suggested by the great similarity between the settings within and between subjects. Figure 23 shows the results averaged across all subjects. In Figure 23a, the mean saturation settings are plotted against the hue value of the comparison filter. The replots of these data in polar coordinates shown in Figure 23b reveals the regular nature of the settings. They appear to lie approximately on an ellipse. That this approximation is quite good can be seen in Figures 23d–23f, which show the data separately for each of the “value-clarity” conditions together with the corresponding fitting ellipse. The parameters of these ellipses are shown in Table 4. The most interesting parameters are the center, the eccentricity, and the size of the ellipse, whereas the absolute size given by \( a \) is mainly determined by the saturation value at the standard hue. The variation of transmittance and clarity had only a relatively small effect on the size and orientation of the fitting ellipses. The most noticeable change is a slight shift of the ellipse center.

### Discussion

The results of Experiment 2 indicate that the shape of iso-saturation curves in the \( H/S \) space is approximately elliptical. This ellipse closely resembles the form and
orientation of the distribution of hue and saturation values found in the simulations reported in the Distribution of filter parameters section.

To be useful in practice, it would be desirable that the shape of iso-saturation curves is invariant with respect to other variables that also influence perceived transparency. Our data suggest that this is the case with respect to variations in transmittance $V$ and clarity $C$. We may also conclude that this also holds with respect to changes in absolute saturation, because otherwise it would not have been possible for the subjects to match saturation at different hues simultaneously along the whole saturation scale.

If iso-saturation curves are really invariant with respect to $S$, then the center of the ellipses would mark the point of zero perceived saturation and should coincide with the zero saturation point $S = 0$. However, we instead found that the center of the ellipses deviate slightly from the zero saturation point $S = 0$, especially in the condition $V = 1$, $C = 1$. These deviations are not artifacts of averaging, because they were also observed in all individual data sets. A possible explanation would be that invariance of the shape of iso-saturation curves under changes of $S$ holds only approximately. This could, in principle, be tested by conducting similar experiments as the one described here but that use low and high saturation subscales in the matching. If invariance holds, then the location and form of the fit ellipses should not be affected by this change in the matching task.

The main limitation in scope of the present experiment is that we only used achromatic backgrounds. It remains quite possible that the shape of the iso-saturation curves is not invariant under changes in the mean color of the background. This would not be problematic, however, as long as the resulting shapes remain sufficiently simple and can be parameterized with statistics of the background color distribution.

Summary and general discussion

A fundamental goal of vision science is to identify and model the representations and computations of internal processes involved in perception, that is, to understand the functional architecture of the visual system (Mausfeld, 2011). These inner processes are not directly accessible and must, therefore, be inferred using known aspects of the perception process as a whole. In the present investigation, we were interested in processes involved in transparency perception, especially in those that use color information. As a heuristic strategy to formulate constraints to the internal processes, we pursued the “filter approach.” In this approach, transparency perception is considered as an attempt of the visual system to detect the presence and relevant properties of optical filters. From this point of view, there are two known parts of the perception process, namely, the input into the visual system, which can be computed from physical properties of optical filters, and an important aspect of the output, namely, the phenomenal impressions that we experience if we look at optical filters. Thus, in this approach, one must use criteria related to these two known parts to constrain and validate models of the computations used in transparency perception.

From the perspective of the filter approach, a criterion that any candidate model has to fulfill is that it allows a successful detection of the relevant distal object. To develop a model with this property, one first needs to identify reliable regularities in the images cast by optical filters that are specific for this class of objects and can, thus, be used by the visual system as a cue. Due to the high complexity of the corresponding image generation process, this is a very difficult task and it thus appears advantageous to consider special cases that result from simplifying assumptions about certain aspects of the scene. This strategy was used in our previous work to identify characteristic color changes caused by optical filters (Faul & Ekroll, 2002). An important additional aspect in the derivation of a suitable model was to consider restrictions on physical parameters typically found in natural scenes, in particular restrictions on illumination, reflectance, and absorption spectra. In this previous work, we also made first steps to validate the derived filter model. We confirmed in computer simulations that it actually describes the color changes caused by optical filters to a good approximation, and experimental tests revealed that it also successfully predicts perceived transparency: The accuracy of the prediction was found to be significantly better than that of the episcotister model.
These encouraging results motivated us to investigate the properties of the model more closely. In the present work, we focused on aspects that could be investigated without substantially changing the scene restrictions made previously. We started in the Robust parameter estimation section by addressing the question of whether the parameters of the model can be estimated in a robust way. Such investigations are useful, because the plausibility of a given model increases if it exhibits a similar robustness behavior as the corresponding perceptual process (including cases where robustness fails). Our analysis demonstrates that unexpected failures of robustness resulting from singularities in the model equations can almost completely be avoided if the estimations in the three color channels are carried out in a coupled way. We also showed that it is possible to use a more global

![Graphs](image-url)

Figure 23. Mean results of all subjects in Experiment 2. (a) The mean settings in each condition in Cartesian coordinates. The vertical dashed line marks the position of the standard hue. (b) The same data in polar coordinates. The red and black boundaries show the maximally realizable saturation due to restrictions of the color space and the monitor gamut, respectively. The oblique dashed line emanating from the center marks the direction of the standard hue. (c) The fitted ellipses for all conditions in comparison. (d–f) For each of the three conditions, the mean settings ±2 SE and an ellipse fitted to the data are shown. An open circle marks the center of the ellipse.

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<td>0.159</td>
<td>0.912</td>
<td>58.4</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0</td>
<td>0.029</td>
<td>0.048</td>
<td>0.368</td>
<td>0.157</td>
<td>0.904</td>
<td>56.7</td>
</tr>
<tr>
<td>1.0</td>
<td>0.5</td>
<td>0.013</td>
<td>0.034</td>
<td>0.373</td>
<td>0.151</td>
<td>0.914</td>
<td>55.2</td>
</tr>
</tbody>
</table>

Table 4. Parameters of the ellipse fitted to the data for three conditions defined by different values for V and C. The columns x and y contain the center position, a and b contain the semi-major and semi-minor lengths, respectively, ε contains the eccentricity, and α contains the orientation of the fitting ellipses. The angle α is measured from the x-axis to the major axis of the ellipse.
estimation procedure based on means and standard deviations in sets of colors that may result either from spatial or temporal integration. This statistical method does not only allow a more robust estimation of the model parameters but has also many additional advantages, for instance, that the proper assignment of four related colors at an X-junction is not necessary, or that isolated colors, like those that are completely covered by the filter, can be integrated in the estimation without any difficulty. However, these advantages can only be brought to bear, if the integration is done over sets of colors that belong to regions with (from the perspective of the model) identical meaning and roughly identical parameter values. It is, at present, an open question how image-based criteria can be used to properly determine such integration regions. It may be interesting to note that Singh and Anderson (2002) used an alternative way to estimate the model parameters from stimuli with many different gray levels (sinusoidal gratings). To reduce the number of gray levels to the four values used in the episcotister model, they selected the minimal and maximal gray levels in the background and the transparent region. However, this approach has two potential drawbacks. A first one is that it is unclear how the selection of maxima and minima can be generalized to three-dimensional color codes, and a second one is that this method does not improve the robustness of the parameter estimation.

The aim of the experiment described in the Comparison of submodels of the filter model section was to select the most appropriate model for further investigations. We compared a “full” and a “reduced” model and presented evidence that the “reduced” submodel in some cases predicts perceived transparency significantly better than the “full” model but that it describes the color changes caused by optical filters less accurately. Thus, computational criteria related to the distal side speak in favor of the “full” model, whereas phenomenal criteria conform better with the “reduced” model. These findings exclude the possibility that the visual system always uses the “full” model in transparency perception. From the perspective of the current approach—that emphasizes the performance aspect of vision—this is somewhat surprising, because the “full” model seems to provide a better basis for transparency detection and is also of similar complexity as the “reduced” model. We discussed several possible interpretations of these findings and experimental strategies to test them. The currently available evidence does not allow to decide between the “full” and “restricted” models and we, therefore, compared both models in further investigations.

In the remainder of the paper, we investigated several aspects of the transparent layer that is inferred from the stimulus. We started by the most obvious one, namely, the perceptual impression that accompanies this inference process, and asked how properties of the perceived transparent layer relate to the parameters of the filter model and ultimately to properties of the distal scene. From this perspective, the filter model may be interpreted as a mediator between the physical and the perceptual world. With respect to the physical side, it approximately describes the image generation process that depends on parameters of the distal object, and with respect to the perceptual side, it determines dimensions of the internal representation of the perceived object. Our original formulation of the model was derived from image generation and the parameters, therefore, have a close relationship to the parameters of optical filters. However, if the filter model actually serves the above-mentioned role, then there must be a different reading of the model parameters that relates them to perceptual dimensions. Observing that the perceived quality of the transparent layer often closely resembles properties of surface colors, we proposed in the Remapping the model parameters to a phenomenological space section a remapping of the transmittance parameter $\tau$ of the original model in terms of “hue,” “saturation,” and (overall) “transmittance” and of the direct reflection parameter $\delta$ in terms of “clarity.” This alternative parameters are understood as properties of an internally represented transparent layer. Thus, the alternative parameters hue, saturation, and (overall) transmittance—just as the transmittance $\tau$ from which they are computed—are not dimensions of a color code and must not be confused with the corresponding dimensions in color space.

The choice of the specific RGB to HSV transform that we used to convert the transmittance parameter $\tau$ to hue, saturation, and (overall) transmittance is by necessity somewhat ad hoc and can only be justified by showing that it yields a good description of empirical observations. A first hint that this is actually the case is provided by the demonstrations in Figures 7–10, which suggest that the alternative parameters actually describe different and intuitively plausible dimensions of the perceived transparent layer. A supporting result with respect to the clarity parameter is that it provides a simple explanation of Singh and Anderson’s (2002, 2006) finding that the ratio of the Michelson contrast in the filter and background regions determined an aspect of perceived transmittance in a matching experiment: As we have shown in the Filter transparency in simple stimuli section, the clarity parameter stands in a close relationship to this measure, and the results of Singh and Anderson (2002) thus suggest the interpretation that the subjects actually matched filter clarity.

In The relation of model parameters to the physical model section, we investigated the alternative parameterization with respect to the distal side. Using computer simulation, we determined how the parameters of the filter model are affected by variations in the thickness and the refractive index of optical filters. The hue parameter was found to be virtually unaffected by these variations, which suggests that it may be used as a rather reliable indicator for the identity of a distal filter object. The estimates of the saturation and the transmittance parameter were found to be closely related to thickness but showed only a very weak dependence on the refractive index. This suggests that both parameters may be used to infer thickness-
related information. This was confirmed in a computer simulation, which showed that the relative filter thickness can be recovered from the spatial distribution of the transmittance and the saturation parameter. This information can, for instance, be used to identify transparent objects based on 3D form information. As expected, the remaining parameter, clarity, was found to depend mainly on the refractive index. Together, these findings suggest that the parameters do not only capture important subjective dimensions of the transparency impression but that they are also intimately related to important properties of the distal object. This adds to the plausibility of the proposed parameterization.

In the new parameterization of $\tau$, the saturation parameter turned out to be more complex than the two other dimensions. The main reason for this is that the maximally attainable saturation value is constrained in complex ways, whereas any value between 0 and 1 may be chosen for the hue and transmittance parameters. In the Restrictions on layer saturation section, we investigated restrictions on possible saturation values resulting from properties of color space. Basically, the question we addressed was: Given a set of background colors and fixed values for the hue, transmittance, and clarity parameters, what is the maximum value possible for the saturation parameter, such that all transformed background colors are still valid, that is, have coordinates inside the color cone? With respect to this question, the case of an achromatic background color is especially interesting, because possible hue/saturation pairs are then isomorphic to the chromaticities of the filtered color. The region of valid hue/saturation pairs for this case is, therefore, numerically identical to the image of a chromaticity diagram in the hue/saturation space. For other cases, however, this isomorphism does not hold. In the context of the filter model, this is not problematic, because it is obvious that the transmittance $\tau$ and the derived alternative dimensions hue, saturation, and (overall) transmittance should not be interpreted as a color code. In the episcotister model, in contrast, the model parameter that determines the color of the perceived transparent layer is understood as a color code (in the prototypical case of a rotating sector disk, it corresponds to the color of the disk surface). This leads to paradoxes, because it was found that stimuli may appear transparent even if the corresponding parameter of the episcotister model is an impossible “color code” outside the color cone. Richards et al. (2009) dubbed such invalid parameters as “imaginary colors.”

Our results further show that the maximum saturation value depends both on the set of background colors and on the value of the hue parameter. Clearly, this has annoying consequences if one wants to construct stimuli conforming to the model and having specific parameter values, because a saturation value suitable for $N$ background colors and a given hue may no longer be possible if an additional background color is included or if the hue value is changed. At first sight, this may appear surprising, because one may ask, why do similar problems not occur with “real optical filters” that in a way also “construct stimuli”? The reason why a real filter always produces valid colors, whereas computing filtered colors from the model may not, is that in the former case we manipulate the light impinging on the cones, but in the latter, we manipulate dimensions of color codes (i.e., the activation of the cones) in an independent way that ignores correlations between color channels that are due to an overlap in their sensitivity curves. It is important to note, however, that these problems are irrelevant from the perspective of the visual system, because the colors in the proximal stimulus are, by definition, always “valid” and the goal is not to construct stimuli but to interpret them. Seen from this point of view, our results merely illustrate that the range of saturation values that may potentially be estimated from a given stimulus can be narrowly reduced if the background colors are highly saturated.

Such highly saturated background colors are, however, highly improbable. This is a consequence of the empirical fact that most illumination and reflection spectra in the environment are smooth and broadband. Thus, a more interesting question is how the filter parameters are distributed given these natural restrictions. In the Distribution of filter parameters section, we used computer simulations to determine frequency distributions for the filter parameters under more realistic conditions. Our results suggest that only relatively small saturation values occur in such cases. In a hue/saturation polar plot, they fall in a small elliptical region around the zero saturation point. Up to this point, we mainly considered nominal parameter values and ignored how these values are related to the perceived properties of the transparent layer. With respect to the transmittance and the clarity parameter, the latter question essentially boils down to the classical problem of determining the subjective scale for an attribute and maximum likelihood difference scaling as proposed by Maloney and Yang (2003) seems to be a promising way to do this. In the present work, we focused on the more complex problem of determining iso-saturation curves, that is, locations in a hue/saturation polar plot that assign to each hue a nominal saturation value such that all corresponding filters are perceived as equally saturated. In this context, the most interesting questions are what shape these curves have and whether and how this shape depends on other filter parameters and properties of the scene. In the experiment reported in the Iso-saturation curves section, we investigated these questions for achromatic background colors and three combinations of the transmittance and clarity parameters. The resulting curves were approximately elliptical in a hue/saturation polar plot and the shape and the orientation of these ellipses closely resemble the distribution of hue/saturation pairs found in the simulation in the Distribution
of filter parameters section. That the curves under each of
the three tested combinations of transmittance and clarity
values are highly similar suggests that the iso-saturation
curves are approximately invariant under variations of
these parameters. It would be interesting to investigate
how the curves are influenced by variations in the
distribution of the background colors and the color of
the illumination. If the curves remain approximately
constant under these variations, then it would potentially
be useful to transform the hue/saturation space in order to
obtain circular iso-saturation curves. In our investigation,
we assumed that the saturation scale is (up to a stretching
of the nominal scale) identical or at least highly similar
for all hues. Whether this is actually true also remains an
open empirical question.

Conclusions and outlook

The purpose of the present paper was, on the one hand,
to demonstrate that the filter approach is a useful heuristic
tool that allows to generate interesting and nontrivial
hypotheses about transparency perception and, on the
other hand, to extend and validate the specific filter model
we proposed in Faul and Ekroll (2002). Naturally, there
remain many open questions. Some possible extensions to
the present approach have already been mentioned above.
A further important problem that is closely related to the
work presented in this paper and that can be investigated
with similar methods is “transparent layer constancy”
(Gerbino et al., 1990). That is, the question to what extent
the estimated parameters of the filter model (that
presumably determine the transparency impression)
remain constant for a fixed distal object under varying
scene properties, in particular varying distributions of the
background colors and changes in illumination. The main
reason why we did not handle this question in the present
due to space limitations. Preliminary experimental
results suggest that the filter parameter estimates are rather
robust under such changes in environmental conditions.

As stressed in the Introduction section, an important
advantage of the filter approach is that it suggests how
research on perceptual transparency that traditionally
mainly focused on simple stimuli may be extended to
more complex situations. A first interesting extension
would be to consider more natural, spatially nonuniform
illuminations that often seem to provide important trans-
parency cues, especially in the case of highly transmissive
media. The specific transparency impression that we have
when looking at a wineglass, for instance, seems to
depend on a large part on properties of the mirror image
of the environment reflected from its surface. From a
formal perspective, such spatially nonuniform illumina-
tions may be incorporated in the filter model by replacing
the constant illumination value in the model with a spatial
illumination map. A further useful extension would be to
consider three-dimensional objects instead of the flat 2D
stimuli traditionally used, because this would allow to
investigate motion- and form-related transparency cues.
The filter approach allows to derive specific predictions
also in these more complex situations. For instance, the
fact that the strength of the mirror reflection from a filter
surface depends on surface orientation (Fresnel effect)
poses the question whether the visual system actually uses
the relation between surface orientation and the strength
of the mirror image as a cue in transparency detection.

Appendix A

Frequency-limited spectra

Frequency-limited spectra were proposed by Stiles,
Wyszecki, and Ohta (1977) to model spectral reflection
properties of real objects. They are defined as
\[ \rho(\lambda) = \frac{1 + \beta(\lambda)}{2}, \]

where

\[ \beta(\lambda) := \sum_{i=-l}^{l} \mu_i \text{sinc}^2[\pi(\omega(\lambda - \lambda_0) - i/2)], \quad (A1) \]

and \(-0.5 \leq \mu_i \leq 0.5\). The essential parameter is the limiting
frequency \(\omega\) that controls the smoothness of the resulting
spectrum: The smaller \(\omega\) is, the smoother the spectrum.
The limiting frequency of natural reflectance spectra has
been found to lie inside the range of 1/100 to 1/50 cycles/nm
(Maloney, 1986). The parameters \(\lambda_0\) and \(l\) are of minor
interest and were, in our simulations, set to \(\lambda_0 = 350\) nm
and \(l = 50\), throughout.

To select a random sample from this collection, one
simply sets all \(\mu_i\) to random values from the interval
\([-0.5, 0.5]\). It is also possible to compute reflectance
spectra, where the reflected light under a given illumination
\(I(\lambda)\) has a specified color code \(C\). Let

\[ b_{ij} := \int_{\lambda} I(\lambda) R_j(\lambda) \text{sinc}^2[\pi(\omega(\lambda - \lambda_0) - i/2)], \quad (A2) \]

where \(R_j(\lambda)\) denotes the sensitivity spectrum of cone class
\(j\), and

\[ \phi_j := \sum_{i=-l}^{l} \mu_i b_{ij}. \quad (A3) \]

The value \(b_{ij}\) is the excitation of cone class \(j\) by basis
function \(i\) under the given illumination and \(\phi_j\) is a
weighted sum of these contributions. Linear programming
is then used to determine $\mu_i$ under the restrictions $-0.5 \leq \mu_i \leq 0.5$ such that $(\phi_L, \phi_M, \phi_S) = X$, with $X := 2C - I$.
Here, $I$ denotes the color code of the illuminant. The frequency-limited spectrum constructed with these $\mu_i$ has the required property.

### Appendix B

**RGB to HSV transform**

MATLAB functions that transform between rgb and hsv coordinates.

```matlab
function hsv = rgbTohsv(r,g,b)
    v = max([r,g,b]); s = (v-min([r,g,b]));
    if s == 0, h = 0;
    else
        if r == v, h = (g-b)/s;
        elseif g == v, h = 2.0+(b-r)/s;
        elseif b == v, h = 4.0+(r-g)/s;
    end
    h = h/6.0; s = s/v;
    if h < 0, h = h+1;
    elseif h > 1, h = 1-h;
    end
    hsv = [h,s,v];
end

function rgb = hsvTorgb(h,s,v)
    if s == 0.0, rgb = [v,v,v];
    else
        if h == 1.0, h = 0.0; end
        h = h * 6;
        f = h - floor(h);
        p = v*(1-s);
        q = v*(1-s*f);
        t = v*(1-s*(1-f));
        switch floor(h)
            case 0, rgb = [v, t, p];
            case 1, rgb = [q, v, p];
            case 2, rgb = [p, v, t];
            case 3, rgb = [p, q, v];
            case 4, rgb = [t, p, v];
            case 5, rgb = [v, p, q];
        end
    end
end
```

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