An object-color space

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Putting aside metaphorical meanings of the term, color space is understood as a vector space, where lights having the same color (i.e., subjectively indistinguishable) are represented as a point. The CIE 1931 color space, empirically based on trichromatic color measurements, is a classical example. Its derivatives, such as CIELAB and sRGB, have been successfully used in many applications (e.g., in color management). However, having been designed for presenting the color of self-luminous objects, these spaces are less suitable for presenting color of reflecting objects. Specifically, they can be used to represent color of objects only for a fixed illumination. Here I put forward a color space to represent the color of objects independently of illumination. It is based on an ideal color atlas comprising the reflectance spectra taking two values: $k$ or $1 - k$ ($0 \leq k \leq 1$), with two transitions (at wavelengths $\lambda_1$ and $\lambda_2$) across the spectrum. This color atlas is complete; that is, every reflecting object is metameric to some element of the atlas. When illumination alters, the classes of metameric reflectance spectra are reshuffled but in each class there is exactly one element of the atlas. Hence, the atlas can uniquely represent the metameric classes irrespective of illumination. Each element of the atlas (thus, object color) is specified by three numbers: (i) $\lambda = (\lambda_1 + \lambda_2)/2$, which correlates well with hue of object color (as dominant wavelength correlates with hue of light color); (ii) $\delta = |\lambda_1 - \lambda_2|$, which correlates with whiteness/blackness; and (iii) $\alpha = |1 - 2k|$, which correlates with chroma of object color (as colorimetric purity correlates with saturation of light color). Using a geographical coordinate system, each element of the atlas (thus, each object color) is geometrically represented as a radius vector so that its length equals $a$, the latitude and longitude being proportional to $\delta$ and $\lambda$, respectively.

Keywords: color, color vision, color space, color coordinates, cone fundamentals, convexity, object-color solid, object-color stimulus, color atlas, metamerism, metamer mismatching, optimal reflectance spectra, spherical chromaticity diagram, chromaticity difference, complementary color


Introduction

There has been a long tradition to represent color geometrically as a point in some space (Kuehni, 2003). Although the term “color space” is often used rather loosely, it has an exact meaning in colorimetry (Brainard, 1995; Schanda, 2007), where color is understood as a class of metameric (i.e., visually indistinguishable) lights that are represented by their spectral power distributions (Koenderink & Doorn, 2003; Krantz, 1975). As these distributions constitute a positive cone in a properly chosen functional space, the classes of metameric lights inherit the linear structure due to Grassmann’s laws (Koenderink & Doorn, 2003; Krantz, 1975; Suppes, Krantz, Luce, & Tversky, 1989). As a result, color can be represented as a point in a vector space. While the validity of Grassmann’s laws remains an open issue (Brill & Robertson, 2007; Logvinenko, 2006), it is widely believed that the physiological basis of metamerism (and Grassmann’s laws) is the equality of the cone photoreceptor outputs. In other words, it is assumed that for two lights to be metameric each type of the photoreceptor cones should respond equally to these lights (Smith & Pokorny, 2003; Stockman & Sharpe, 2007; Wyszecki & Stiles, 1982).

More formally, let us define the color signal in response to a light with the spectral power distribution $I(\lambda)$ as a triplet $(\varphi_1(I), \varphi_2(I), \varphi_3(I))$, where

$$\varphi_i(I) = \int_{\lambda_{min}}^{\lambda_{max}} I(\lambda) t(\lambda) p_i(\lambda) d\lambda,$$

(1)

$t(\lambda)$ is the transmittance spectrum of the ocular media (e.g., the lens and macular pigment), and $p_i(\lambda)$ is the spectral absorption of the $i$th photopigment, ($i = 1, 2, 3$), that is assumed to be positive throughout the visible spectrum interval $[\lambda_{min}, \lambda_{max}]$. I will use throughout the paper the following values: $\lambda_{min} = 380$ nm, $\lambda_{max} = 780$ nm. Two lights $I_1(\lambda)$ and $I_2(\lambda)$ are assumed to be metameric if and only if the color signals for these lights are equal, that is, $\varphi(I_1) = \varphi(I_2)$ for each $i$.

Following the established terminology (e.g., Judd & Wyszecki, 1975; Stockman & Sharpe, 2007), the product

$$s_i(\lambda) = t(\lambda) p_i(\lambda),$$

(2)

will be referred to as the $i$th cone fundamental. The notations $S, M,$ and $L$ will be used for the response of the
cone fundamentals with the peak sensitivity in the short-, middle-, and long-wave range of the visible spectrum, respectively.

When one represents a color signal as the point in the 3D space with the Cartesian coordinates \((S, M, L)\), the color signals from all the lights form a convex cone in this space (referred to as the \textit{color signal cone}). Each color signal represents a class of metameric lights. As metameric lights are assumed to have the same color, this cone can be considered as a geometrical representation of color in the 3D space that will be referred to as the \textit{SML color space}.

Although, in fact, such a color space was designed for representing color of lights (i.e., self-luminous objects), it has also been used to represent the color of reflecting objects. Specifically, a surface with the spectral reflectance function \(x(\lambda)\) illuminated by a light with the spectral power distribution \(I(\lambda)\) produces a color signal the \(i\)th component of which is

\[
\varphi_i(x) = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} x(\lambda)I(\lambda)s_i(\lambda)d\lambda.
\]

It seems safe to assume that when the illumination is fixed (i.e., all the objects are lit by the same light), reflecting objects producing the same color signal (referred to as \textit{metameric reflecting objects}) will appear as having the same color. Therefore, under constant illumination each color signal will represent a class of metameric spectral reflectance functions. The color signals produced by all the reflecting objects under the same illuminant form a closed convex volume (referred to as the \textit{object-color solid}) in the color signal cone (Koenderink & Doorn, 2003; Luther, 1927; Maximov, 1984; Nyberg, 1928; Schrodinger, 1920/1970; Wyszecki & Stiles, 1982).

At the first glance, the object-color solid seems to provide the geometrical representation of object color in the \textit{SML} color space the same way as the color signal cone does for light color. However, the analogy is rather superficial because such a representation depends on the illuminant. When the illuminant alters, metameric spectral reflectance functions might cease to be metameric. This phenomenon is known as \textit{metamer mismatching} (Wyszecki & Stiles, 1982). Conversely, spectral reflectance functions that produce different color signals under one illuminant may become metameric under the other. So, after an illumination change, some spectral reflectance functions will remain in the same metameric class while others will fall into different classes. Therefore, the reflecting object metamersism as defined above is different for different illuminants.

Because of metamer mismatching, there is no natural way to establish a correspondence between the metameric classes produced under one illumination and those produced under another. In other words, there is no natural one-to-one relationship between two object-color solids induced by different illuminants. To show what is meant by natural relationship, consider a particular case when there is such. Specifically, restrict ourselves to the boundaries of the two object-color solids obtained for different illuminants.

While for every spectral reflectance function mapping into the object-color solid interior, there is an infinite number of metameric spectral reflectance functions, there is no metamerism on the object-color solid boundary.\(^1\) To be more exact, there is no spectral reflectance function metameric to the so-called \textit{optimal} spectral reflectance functions, that is, those which map to the boundary of the object-color solid. For example, there are infinitely many spectral reflectance functions mapping to the center of the object-color solid, one of them being a function \(x(\lambda) \equiv 0.5\), which takes 0.5 at every wavelength within the visible spectrum (written \(x_{0,5}(\lambda)\)). However, only one spectral reflectance function maps to the north (respectively, south) pole of the object-color solid, namely, the “perfect reflector” \(x(\lambda) \equiv 1\) (respectively, the “ideal black” \(x(\lambda) \equiv 0\), which takes 1 (respectively, 0) at every wavelength within the visible spectrum.

It seems natural to map one object-color solid boundary onto the other as follows. Let \((S, M, L)\) be a point on the boundary of the object-color solid obtained for the illuminant \(I(\lambda)\). There is just one optimal spectral reflectance function that produces this color signal. Denote it \(x_{\text{opt}}(\lambda)\). As the set of optimal spectral reflectance functions remains the same for a very broad class of illuminants (see \textit{Optimal spectral reflectance functions and object-color solid section}), \(x_{\text{opt}}(\lambda)\) will map to some point in the boundary of the object-color solid obtained for the other illuminant, \(I(\lambda)\), say, \((S', M', L')\). A one-to-one map \((S, M, L) \rightarrow (S', M', L')\) establishes natural correspondence between the object-color solid boundaries because it puts in correspondence color signals which represent the same classes of metamerism (though, they are singletons under both the illuminants). Note that this map is non-linear.

Extending this map to the interiors of the object-color solids results in a many-to-many map because of metamer mismatching. Yet, if we replace each metameric class with just one of its members, it becomes possible. Specifically, a one-to-one relationship between the classes of metameric reflectances produced by different illuminants can be established through a color atlas.\(^2\)

In the present context, any sample of non-metameric spectral reflectance functions will be referred to as a \textit{color atlas}. Each element of a color atlas represents a metameric class to which it belongs. A color atlas that represents all the classes of metamerism will be called \textit{complete}. As every optimal spectral reflectance function makes a metameric class on its own, a complete color atlas includes all the optimal spectral reflectance functions.

Being a color atlas under one illumination, a set of spectral reflectance functions may cease being a color
atlas under the other illumination because of metamer mismatching. Therefore, when using a notion of color atlas, one has always to specify an illuminant for which the given set of spectral reflectance functions is a color atlas. When a sample of spectral reflectance functions is a color atlas for the whole family of illuminants, we will say that the color atlas is invariant with respect to this illuminant family. To summarize, given a family of illuminants, a complete illuminant-invariant color atlas is a set of spectral reflectance functions such that (i) each class of metamerism contains exactly one element of this set under any illuminant in the family, and (ii) any two spectral reflectance functions from the set do not become metamer under any illuminant in the family.

As there is a one-to-one map between an illuminant-invariant color atlas and a set of classes of metamerism under any illuminant (from the family in question), a complete illuminant-invariant color atlas induces a one-to-one map between the classes of metamerics (thus, between the color solids) induced by different illuminants. Indeed, let us put in correspondence those metamer classes that contain the same element of the color atlas.

It follows that a complete color atlas, invariant with respect to some illuminant family, uniquely represents all the metamer classes, thus, all the object colors under any illuminant in this family. Therefore, any spatial representation of such an atlas will be also a spatial representation of the object colors for any illuminant in this family.

In this article, I introduced a particular complete color atlas invariant with respect to illuminants with positive spectral power distribution (Illuminant invariant color atlas section). The Object-color space section shows (i) that this color atlas can be geometrically represented as the unit ball in a 3D space and (ii) that this space can be used for representing all the reflecting objects illuminated by lights with positive spectral power distribution. Such representation will be independent of illumination in the sense that the coordinate system in the 3D space will remain the same for the illuminations in question.

A complete illuminant-invariant color atlas can be used to investigate how the color of a particular object (represented by its spectral reflectance function) transforms with illumination. The problem of color transformation induced by illumination can be reduced to the following two. Firstly, the color atlas itself may change its color appearance when the illumination alters. For example, while a white surface remains clearly recognizable as white under various chromatic illuminations, its appearance under, say, yellow light differs from that under blue light. This is a matter of experimental investigation that goes beyond the scope of the present article. Secondly, the color of an object may change because the object may move from one class of metamerism to the other because of an illumination change. As a result, the object will become metameric to a different element of the color atlas. Such an illuminant-induced color stimulus shift is discussed in the Illuminant- and observer-induced color stimulus shifts section.

### Illuminant invariant color atlas

#### Optimal spectral reflectance functions and object-color solid

The optimal spectral reflectance functions can take only two values: 0 or 1 (Schroedinger, 1920). For example, the spectral reflectance function \( x(\lambda) = 1 \) is optimal. Another example of spectral reflectance functions that can be optimal is a step function (i.e., a piecewise constant function taking 1 or 0) with transition at the wavelength \( \lambda_1 \):

\[
x_1(\lambda; \lambda_1) = \begin{cases} 0, & \text{if } \lambda_{\min} \leq \lambda < \lambda_1; \\ 1, & \text{if } \lambda_1 \leq \lambda \leq \lambda_{\max}. 
\end{cases}
\]

(4)

Note that \( x(\lambda) = 1 \) is a particular case of \( x(\lambda; \lambda_1) \) when \( \lambda_1 = \lambda_{\min} \). Given \( \lambda_2 \geq \lambda_1 \), the step function with transition wavelengths \( \lambda_1 \) and \( \lambda_2 \)

\[
x_2(\lambda; \lambda_1, \lambda_2) = x_1(\lambda; \lambda_1) - x_1(\lambda; \lambda_2),
\]

(5)

can also be an optimal spectral reflectance function. Equation 5 will be referred to as the rectangular spectral reflectance function. Note that \( x_1(\lambda; \lambda_1) \) is a particular case of \( x_2(\lambda; \lambda_1, \lambda_2) \) when \( \lambda_2 = \lambda_{\max} \). Generally, the step function with transitions at the wavelengths \( \lambda_{\min} < \lambda_1 < \lambda_2 < \ldots < \lambda_m < \lambda_{\max} \)

\[
x_m(\lambda; \lambda_1 \ldots \lambda_m) = \sum_{i=1}^{m} (-1)^{i-1} x_1(\lambda; \lambda_i),
\]

(6)

can be an optimal spectral reflectance function.

For any optimal spectral reflectance function \( x_{\text{opt}}(\lambda) \), the spectral reflectance function \( 1 - x_{\text{opt}}(\lambda) \) has proven to be also optimal. Particularly, if \( x_m(\lambda; \lambda_1, \ldots, \lambda_m) \) is an optimal spectral reflectance function, the spectral reflectance function \( 1 - x_m(\lambda; \lambda_1, \ldots, \lambda_m) \) is also optimal. A pair of optimal spectral reflectance functions \( x_{\text{opt}}(\lambda) \) and \( 1 - x_{\text{opt}}(\lambda) \) will be called complementary. The optimal spectral reflectance functions \( x(\lambda) = 1 \) and \( x_m(\lambda; \lambda_1, \ldots, \lambda_m) \) for any integer \( m \) will be referred to as of type I, the optimal spectral reflectance functions \( x(\lambda) = 0 \) and \( 1 - x_m(\lambda; \lambda_1, \ldots, \lambda_m) \) as of type II.

There is a general belief (Koenderink & Doorn, 2003; MacAdam, 1935; Schroedinger, 1920; Wyszecki & Stiles, 1982) that the optimal spectral reflectance functions are step functions with not more than two transitions across
the visible spectrum. (It will be referred to as the two-transition assumption). However, this is not, strictly speaking, true. In fact, the number of transitions depends on the shape of the cone fundamentals (Maximov, 1984; West & Brill, 1983). More specifically, a theorem has been proved (Logvinenko & Levin, 2009) from which it follows that, for continuous cone fundamentals \( s_1(\lambda), s_2(\lambda), \) and \( s_3(\lambda) \), and an illuminant with integrable spectral power distribution \( I(\lambda) \), if \( \lambda_1, \ldots, \lambda_m \) are the only roots of the following equation

\[
k_1s_1(\lambda)I(\lambda) + k_2s_2(\lambda)I(\lambda) + k_3s_3(\lambda)I(\lambda) = 0,
\]

where \( k_1, k_2, \) and \( k_3 \) are arbitrary real numbers (at least one of which is not equal to zero), then a step function (Equation 6) with transitions at the wavelengths \( \lambda_1, \ldots, \lambda_m \) will be an optimal spectral reflectance function.

If the illuminant is such that \( I(\lambda) > 0 \) for \( \lambda_{\min} \leq \lambda \leq \lambda_{\max} \), Equation 7 is equivalent to the following

\[
k_1s_1(\lambda) + k_2s_2(\lambda) + k_3s_3(\lambda) = 0.
\]

Therefore, the shape of the illuminant spectral power distribution does not affect the set of optimal spectral reflectance functions unless it takes zero values within the visible spectrum interval.

The roots of Equation 8 have a simple geometrical meaning. Recall that the color signal of the monochromatic light with the wavelength \( \lambda \) is given by \( (s_1(\lambda), s_2(\lambda), s_3(\lambda)) \). When \( \lambda \) runs through the interval \([\lambda_{\min}, \lambda_{\max}]\), a point \((s_1(\lambda), s_2(\lambda), s_3(\lambda))\) makes a curve in the SML color space which is usually referred to as the spectral curve (Figure 1). The cone through the spectral curve is referred to as the spectral cone (Figure 1). It represents color of the monochromatic lights. The color signal cone proves to be the convex hull of the spectral cone. The roots of Equation 8 are the points where the spectral curve intersects the plane defined by the equation \( k_1S + k_2M + k_3L = 0 \).

The shape of a projection of the spectral curve to any plane not containing the origin is indicative of the maximum number of transitions for the optimal reflectance functions. For example, let us choose a plane not containing the origin and consider a contour made by the spectral curve projection to this plane (referred to as the spectrum locus). Let us complete this contour with the interval joining the ends of the spectrum locus (the purple interval). If the resultant contour (referred to as the completed spectral contour) is convex, then, as noted by West and Brill (1983), the two-transition assumption holds true.

As it is safe to assume that the transmittance spectrum of the ocular media is everywhere positive on \([\lambda_{\min}, \lambda_{\max}]\) (i.e., \( \tau(\lambda) > 0 \) in Equation 2), Equation 8 amounts to

\[
k_1p_1(\lambda) + k_2p_2(\lambda) + k_3p_3(\lambda) = 0.
\]

Therefore, the lens and macular pigment do not affect the set of optimal spectral reflectance functions. Generally, any pre-receptor filter with positive transmittance spectrum cannot affect the set of optimal spectral reflectance functions. It is fully determined by the photopigment spectral absorption.

The absorbance spectra of the cone photopigments are known to be described by a relatively simple analytical expression (Lamb, 1999). The spectral sensitivities of the three cone photopigments calculated from the photopigment optical density template put forward by Govardovskii with coauthors (Govardovskii, Fyhrquist, Reuter, Kuzmin, & Donner, 2000) are presented in Figure 2. Note that although the spectral responses, especially the \( S \)-cone’s, are very low in the long wavelength end of the spectrum, they differ from zero. (This property is of great theoretical importance. For instance, letting \( p_1(\lambda) = 0 \) for \( \lambda_{\min} > \lambda' \geq \lambda \geq \lambda_{\max} \) in Equation 5, will result in metamerism in the object-color solid boundary.) The cone fundamentals based on these photopigment spectra are presented in Figure 3.

The completed spectral contour in the unit plane of the SML space based on these cone fundamentals is shown in Figure 4. As one can see, it is not convex. Firstly, the short wavelength end of the spectrum locus makes a well-pronounced “beak.” Secondly, at the opposite end the spectrum locus reverses its direction making a self-intersection. This results in a so-called hue-reversal effect described by Brindley (1955). All this leads to that the purple interval lies inside the color signal cone.

Because of concavity of the completed spectral contour in Figure 4, there exist many lines intersecting the spectrum locus in Figure 4 at more than two points. A plane through any of these lines and the origin will intersect the spectral curve at more than two points. Therefore, the concavity of the completed spectral contour...
entails the existence of optimal spectral reflectance functions with more than two transitions, thus the failure of the two-transition assumption.

It follows from Equation 8 that any three linear independent functions that are the linear transformation of the cone fundamentals will determine the same set of optimal spectral reflectance functions as the cone fundamentals themselves. The color matching functions are generally believed to be the linear transformation of the cone fundamentals (e.g., Judd & Wyszecki, 1975; Stockman & Sharpe, 2007). If this were the case, then the optimal spectral reflectance functions derived from the color matching functions should include spectral reflectance functions with more than two transitions. However, the completed spectral contour (in the unit plane) derived from the color matching functions adopted by the CIE as the standard colorimetric observer (Figure 5) is convex. This indicates that for this observer the two-transition assumption holds true.

The completed spectral contour plotted for the cone fundamentals proposed by DeMarco, Pokorny, and Smith (1992) is also convex (Figure 6). The completed spectral
contour based on the cone fundamentals developed by Stockman and Sharpe (2000) is not, strictly speaking, convex (Figure 7). A departure from convexity similar to that in Figure 4 takes place at the long-wavelength end of the spectrum locus. However, no beak at the short-wavelength end of the spectrum locus is present in Figure 7. It follows that many optimal spectral reflectance functions with the third and fourth transitions at the short-wavelength end of the visible spectrum interval will be lost when using these cone fundamentals.

Therefore, contrary to the view that the different sets of the cone fundamentals put forward by various authors differ from each other mainly by the spectral form of the pre-receptor filter (Smith, Pokorny, & Zaidi, 1983), there are essential differences between them. More specifically, if the major difference between them were the pre-receptor filter transmittance, the spectrum loci determined by them would have been either all convex or all concave. However, as follows from Figures 5–7, this is not the case. Furthermore, because of the qualitative difference between the optimal spectral reflectance set induced by the cone fundamentals based on the photopigment spectra and that derived from the color matching functions, it is unlikely that the latter can be represented as a linear transform of the former. Therefore, if Govardovskii’s template describes accurately enough the photopigment absorption in human cones, then perhaps the decision about match in the color matching experiments is related to the cone outputs in a more complicated way than hitherto assumed.

As the two-transition assumption is taken for granted in the color literature, of great theoretical and practical importance is how large is the difference between the real object-color solid and that obtained under the two-transition assumption. Specifically, let us call the volume in the SML color space made by the step functions with not more than two transitions the regular object-color solid (Figure 8). It is a closed volume nested into the real object-color solid.

The curves in Figure 8 are the lines of constant \( \lambda_1 \) and \( \lambda_2 \). Specifically, each such line represents the color signals induced by the spectral reflectance functions \( x_2(\lambda; \lambda_1, \lambda_2) \) (Equation 5) when either \( \lambda_1 \) or \( \lambda_2 \) is fixed. Two coordinate lines, corresponding to the two particular cases when either \( \lambda_1 = \lambda_{\text{min}} \) or \( \lambda_2 = \lambda_{\text{max}} \), will be referred to as the meridians of the regular object-color solid. When \( \lambda_2 = \lambda_{\text{max}} \), the meridian (referred to as the main meridian) is an image of the spectral reflectance functions \( x_1(\lambda; \lambda_1) \) (Equation 4). When \( \lambda_1 = \lambda_{\text{min}} \), the meridian (referred to as the opposite meridian) is an image of the optimal spectral reflectance functions \( 1 - x_1(\lambda; \lambda_2) \). The main and opposite meridians are marked with red and blue, respectively, in Figure 2, the main meridian being almost
hidden. The space between the coordinate lines corresponds to \((\lambda_{\text{max}} - \lambda_{\text{min}}) / 30\). As can be seen, the coordinate lines are not evenly distributed across the boundary surface. They can be made equally spaced after reparameterizing the wavelength interval (see Wavelength reparameterization section).

It turns out that the maximal difference between the real and regular object-color solids is very small (see Optimal metamers and optimal color atlas section). Therefore, the regular object-color solid, which is a great deal easier to evaluate from the computational point of view, can be taken as an approximation to the real object-color solid in many practical applications.

**Optimal metamers and optimal color atlas**

Given an illuminant \(I(\lambda) > 0\), for each spectral reflectance function \(x(\lambda)\) there is (i) a unique optimal spectral reflectance function \(x_{\text{opt}}(\lambda)\) and (ii) a unique number \(0 \leq a \leq 1\) such that spectral reflectance function

\[
(1 - a)x_{0.5}(\lambda) + ax_{\text{opt}}(\lambda),
\]

is metameric to \(x(\lambda)\). (Recall that \(x_{0.5}(\lambda) \equiv 0.5\).

The set of all the spectral reflectance functions (Equation 10) is a complete color atlas. Indeed, given an arbitrary \(x(\lambda)\), the optimal spectral reflectance \(x_{\text{opt}}(\lambda)\) that determines the spectral waveform of (Equation 10) can be found from the condition that the color signal \(\varphi_1(x_{\text{opt}}), \varphi_2(x_{\text{opt}}), \varphi_3(x_{\text{opt}})\) is the boundary point of the object-color solid lying on the same radius as \(\varphi_1(x), \varphi_2(x), \varphi_3(x)\).

As the spectral waveform of (Equation 10) is fully specified by the optimal spectral reflectance functions \(x_{\text{opt}}(\lambda)\), this atlas will be referred to as the optimal color atlas. A member (Equation 10) of the optimal color atlas metameric to \(x(\lambda)\) will be called the optimal metamer of \(x(\lambda)\). It takes two values: 0.5(1 ± \(a\)), the difference between which being \(a\). Therefore, \(a\) indicates how much the spectral reflectance function \((1 - a)x_{0.5}(\lambda) + ax_{\text{opt}}(\lambda) = x_{0.5}(\lambda) + a(x_{\text{opt}}(\lambda) - x_{0.5}(\lambda))\) deviates from the level of 0.5. I will call the quantity \(a\) the chromatic amplitude of the optimal metamer (Equation 10). It can be evaluated as

\[
a = \frac{\sum_{j=1}^{3}(\varphi_j(x) - \varphi_j(x_{0.5}))^2}{\sum_{j=1}^{3}(\varphi_j(x_{\text{opt}}) - \varphi_j(x_{0.5}))^2}.
\]

When chromatic amplitude \(a\) goes from 0 to 1, the color signal of Equation 10 moves from the center to the boundary of the object-color solid remaining on the same radius.

The optimal color atlas proves to be invariant with respect to illuminants with positive spectral power distribution (see Appendix A2). It should be kept in mind that the optimal color atlas invariance does not mean that the optimal metamer of a particular spectral reflectance should be illuminant independent. In fact, many spectral reflectance functions have different optimal metamers under different illuminants. An alteration of the optimal metamer caused by an illuminant change reflects a shift of this reflectance from one class of metamerism to the other because of the illuminant change. In other words, an alteration of the optimal metamer for a particular object indicates the objective alteration of the color of this object (see Illuminant-induced color stimulus shift section).

Like optimal spectral reflectance functions, the optimal metamers will be divided into two groups—of types I and II—depending on shape of their spectral waveform. Specifically, the optimal metamer (Equation 10) will be assigned the same type as the optimal reflectance \(x_{\text{opt}}(\lambda)\) (see Optimal spectral reflectance functions and object-color solid section). Generally, a spectral reflectance function will be referred to as of type I (respectively, II) if its optimal metamer is of type I (respectively, II).

For each optimal metamer \(x(\lambda) \neq x_{0.5}(\lambda)\), there is a unique optimal metamer that differs from \(x(\lambda)\) only in type. Two such optimal metamers will be called complementary. They map to the points symmetrical with respect to the object-color solid center. For the sake of generality, the optimal metamer \(x_{0.5}(\lambda)\) will be assumed to be complementary to itself. The colors of two complementary optimal metamers will be called complementary. For each object color, there is only one complementary color.

As the object-color complementarity has been defined in terms of the optimal color atlas, it is independent of illumination. Note, however, that two spectral reflectance functions that have complementary colors under one illuminant may prove to have non-complementary colors under the other illuminant (see Illuminant-induced color stimulus shift section).

Denoting \(\tilde{x}_{\text{opt}}\), the optimal spectral reflectance function complementary to an optimal spectral reflectance function \(x_{\text{opt}}\) (i.e., \(x_{\text{opt}} = 1 - x_{\text{opt}}\)), we get \(x_{0.5}(\lambda) + a(x_{\text{opt}}(\lambda) - x_{0.5}(\lambda)) = x_{0.5}(\lambda) - a(x_{\text{opt}}(\lambda) - x_{0.5}(\lambda))\). This equation allows us to extend formally the chromatic amplitude range treating an optimal metamer with negative chromatic amplitude as that of positive chromatic amplitude of the same magnitude and complementary spectral waveform. For example, the optimal spectral reflectance function \(x(\lambda) \equiv 0\) can be interpreted as the optimal spectral reflectance function \(x(\lambda) \equiv 1\) with negative chromatic amplitude −1. Hence, using negative chromatic amplitude, we can restrict our consideration to only optimal spectral reflectance functions of type I. In this case, the spectral reflectance functions of negative chromatic amplitude will be exactly the set of the spectral reflectance functions of type II. I will use the term purity for the absolute value of chromatic amplitude.

The optimal color atlas is, generally, different for different cone fundamentals (thus, color matching functions) because it is determined by the set of the optimal
object-color stimuli (i.e., the optimal spectral reflectance functions). For example, when the two-transition assumption holds true, the optimal object-color stimuli are the rectangular spectral reflectance functions. When the two-transition assumption fails there are optimal spectral reflectance functions with more than two transitions. Therefore, in this case the optimal color atlas also includes spectral reflectance functions with more than two transitions.

As the set of the optimal object-color stimuli, in fact, generates the optimal color atlas, it will play an important role in our analysis. I will refer to the set of the optimal object-color stimuli as the *chromatic base* of the optimal color atlas. It will be shown in the Perceptual correlates of the ad color descriptors section that the chromatic base specifies the object-color hues. In the case of the two-transition assumption, the chromatic base comprises only rectangular spectral reflectance functions. It will be called the *rectangle chromatic base*.

It is important to note that even when the two-transition assumption fails the rectangle chromatic base can be used to represent the optimal color atlas. More specifically, given a spectral reflectance function \( x(\lambda) \) producing the color signal \((\varphi_1(x), \varphi_2(x), \varphi_3(x))\) (see Equation 3), consider simultaneous equations

\[
\int_{x_{\min}}^{x_{\max}} (x_{0.5}(\lambda) + \alpha(x_2(\lambda; \lambda_1, \lambda_2) - x_{0.5}(\lambda))) \cdot I(\lambda)s(\lambda)d\lambda = \varphi(x),
\]

where \( x_2(\lambda; \lambda_1, \lambda_2) \) is a rectangular spectral reflectance function with the transition wavelengths \( \lambda_1 \) and \( \lambda_2 \) (Equation 5) \((i = 1, 2, 3)\). These equations can always be resolved with respect to \( \alpha, \lambda_1, \) and \( \lambda_2 \). If the color signal \((\varphi_1(x), \varphi_2(x), \varphi_3(x))\) lies within the regular object-color solid, the solution will be such that \( |\alpha| \leq 1 \). For those color signals that are between the boundary surfaces of the real and regular object-color solids, we will get \( |\alpha| > 1 \). In this case \((1 + \alpha) / 2 \) exceeds 1, and \((1 - \alpha) / 2 \) is negative. While such profiles cannot be treated as spectral reflectance functions, I will consider them as improper spectral reflectance functions. Taking them into consideration allows one to represent the optimal color atlas by rectangle spectral reflectance functions with not more than two transitions.

So, even when the two-transition assumption is not satisfied, for each spectral reflectance function \( x(\lambda) \), there is a (perhaps, improper) spectral reflectance function \( x_{0.5}(\lambda) + \alpha(x_2(\lambda; \lambda_1, \lambda_2) - x_{0.5}(\lambda)) \) that meets Equation 12. It will be called the *rectangular metamer* of the object-color stimulus \( x(\lambda) \). The rectangular metamer will be referred to as improper if \( |\alpha| > 1 \). Although the set of all the rectangle metamers for all the object-color stimuli cannot be a proper colors atlas (as it contains improper spectral reflectance functions), I will refer to it as the *rectangular color atlas*. To avoid confusion, I will call improper those color atlases that contain improper spectral reflectance functions. As a rectangular metamer can be specified by only three numbers \((\lambda_1, \lambda_2, \) and \( \alpha)\), using the rectangle color atlas will allow a three-coordinate representation of the optimal color atlas.

There are, of course, many other improper color atlases that provide a three-coordinate representation of the optimal color atlas. For example, given three spectral reflectance functions (which do not metameric under any illuminant with positive spectral power distribution) \( x_1(\lambda), x_2(\lambda), \) and \( x_3(\lambda) \), any element of the optimal color atlas \( x(\lambda) \) is metameric to some linear combination of these functions: \( k_1x_1(\lambda) + k_2x_2(\lambda) + k_3x_3(\lambda) \). The weights \( k_1, k_2, \) and \( k_3 \) can be considered as linear coordinates of the element of the optimal color atlas. When \( x(\lambda) \) runs over the whole optimal color atlas, the resultant set of the functions \( \{k_1x_1(\lambda) + k_2x_2(\lambda) + k_3x_3(\lambda)\} \), makes an improper color atlas that will be referred to as a *linear color atlas* based on the basis functions \( x_1(\lambda), x_2(\lambda), x_3(\lambda) \).

It should be borne in mind that a coordinate representation of the optimal color atlas based on an improper color atlas depends, generally, on both illumination and cone fundamentals. Specifically, a linear color atlas, strictly speaking, will provide different coordinates for an optimal object-color stimuli under different illuminations unless this stimulus is a basis function. Direct calculations show that the variation of the linear coordinates, for example, for a perfect reflector can be quite large even for natural illuminants. Therefore, linear color atlases do not, generally, provide the coordinate representation of the optimal color atlas that is constant with respect to illumination.

While the three-coordinate representation of the optimal color atlas based on the rectangle color atlas also depends on illuminant, this dependence proves to be very small. More specifically, the set of optimal spectral reflectance functions has, firstly, been evaluated for the cone fundamentals based on the photopigment spectra (Figure 3). Many of rectangular spectral reflectance functions (with two transitions) have proved to be the optimal object-color stimuli. For these, the coordinates \( \lambda_1, \lambda_2, \) and \( \alpha \) are, obviously, independent of illumination. Still, a number of optimal spectral reflectance functions with three and four transitions have been found. These spectral reflectance functions are, generally, metameric to different rectangular metamers under different illuminants. The rectangle metamers have been calculated for the optimal spectral reflectance functions with more than two transitions for a few illuminants. Their purity has been found to fall into the narrow band between 1 and 1.01. Hence, if one measures the distance between the boundaries of the real and regular object-color solids along radii, then the maximum distance does not exceed 1%. It follows that the regular object-color solid is a good approximation to the real object-color solid. Furthermore, the variability of the coordinates \( \lambda_1, \lambda_2, \) and \( \alpha \) for the optimal spectral reflectance functions with more than two transitions has been found to be negligibly small. Therefore, while being of theoretical importance, the dependence of the coordinates \( \lambda_1, \lambda_2, \) and \( \alpha \) on illuminant is unlikely to have
a significant impact in most practical applications. Thus, the rectangle color atlas can be used to provide the three-coordinate representation of the optimal color atlas which is quite easy to compute.

**Rectangle color atlas and rectangular metamers**

The rectangle color atlas is determined by the set of rectangular spectral reflectance functions. Each such function is completely specified by the transition wavelengths, λ₁ and λ₂, and its type. If it is of type I, it takes up 1 within the interval [λ₁, λ₂], its complementary spectral reflectance function taking up 1 exactly outside it. The interval [λ₁, λ₂] can be specified by its width δ = |λ₁ − λ₂| and the position of its center \( \bar{\lambda} = (\lambda_1 + \lambda_2)/2 \). I refer to δ as the spectral bandwidth and \( \bar{\lambda} \) as the central wavelength for a rectangular spectral reflectance function of type I. Thus, the rectangular metamer of a spectral reflectance function of type I is completely specified by the three numbers: purity, spectral bandwidth, and central wavelength, \( \bar{\lambda} \). See text.

Direct calculation shows that in this case (i.e., for rectangular spectral reflectance functions of type II), the spectral bandwidth is

\[
\delta = (\lambda_{\text{max}} - \lambda_{\text{min}}) - |\lambda_1 - \lambda_2|,
\]

and the central wavelength is given by

\[
\bar{\lambda} = \begin{cases} 0.5((\lambda_1 + \lambda_2) + (\lambda_{\text{max}} - \lambda_{\text{min}})), & \text{if } (\lambda_1 + \lambda_2) < (\lambda_{\text{max}} - \lambda_{\text{min}}), \\ 0.5((\lambda_1 + \lambda_2) - (\lambda_{\text{max}} - \lambda_{\text{min}})), & \text{if } (\lambda_1 + \lambda_2) > (\lambda_{\text{max}} - \lambda_{\text{min}}). \end{cases}
\]

It follows that the central wavelengths of complementary rectangular spectral reflectance functions lie on the opposite ends of the diameter of the visible spectrum circle, the sum of their spectral bandwidths being equal to \( \lambda_{\text{max}} - \lambda_{\text{min}} \). This is in line with our intuition of complementarity.

The rectangular spectral reflectance function \( x(\lambda) \equiv 0 \) has zero spectral bandwidth; however, central wavelength is not defined for it. Likewise, for \( x(\lambda) \equiv 1 \), the spectral bandwidth \( \delta = \lambda_{\text{max}} - \lambda_{\text{min}} \), the central wavelength being not defined.

Conversely, a spectral bandwidth and a central wavelength uniquely determine the rectangular spectral reflectance function. Specifically, given \( 0 < \delta < (\lambda_{\text{max}} - \lambda_{\text{min}}) \) and \( \lambda_{\text{min}} < \bar{\lambda} < \lambda_{\text{max}} \), a pair \( (\delta, \bar{\lambda}) \) uniquely defines the rectangular spectral reflectance function, the transition wavelengths \( \lambda_1 \) and \( \lambda_2 \) of which are given by

\[
\begin{align*}
\lambda_1 &= \begin{cases} \bar{\lambda} - 0.5\delta, & \text{if } (\bar{\lambda} - 0.5\delta) > \lambda_{\text{min}}; \\ \bar{\lambda} - 0.5\delta + (\lambda_{\text{max}} - \lambda_{\text{min}}), & \text{if } (\bar{\lambda} - 0.5\delta) < \lambda_{\text{min}}; \end{cases} \\
\lambda_2 &= \begin{cases} \bar{\lambda} + 0.5\delta, & \text{if } (\bar{\lambda} + 0.5\delta) < \lambda_{\text{max}}; \\ \bar{\lambda} + 0.5\delta - (\lambda_{\text{max}} - \lambda_{\text{min}}), & \text{if } (\bar{\lambda} + 0.5\delta) > \lambda_{\text{max}}. \end{cases}
\end{align*}
\]
By definition, zero spectral bandwidth ($\delta = 0$) defines the optimal spectral reflectance function $x(\lambda) \equiv 0$. Likewise, the spectral bandwidth $\delta = \lambda_{\text{max}} - \lambda_{\text{min}}$ defines the optimal spectral reflectance function $x(\lambda) \equiv 1$.

Any pair ($\delta$, $\lambda$) defines a unique location on the regular object-color solid boundary surface. The rectangular spectral reflectance functions with the same central wavelength map to a closed curve on this surface. So do the rectangular spectral reflectance functions with the same spectral bandwidth. These curves can serve as the coordinate web on the regular object-color solid boundary surface (Figure 11).

To summarize, the rectangular metamer of a spectral reflectance function is specified by the three numbers: purity, spectral bandwidth, and central wavelength. I will call the purity, spectral bandwidth, and central wavelength of the rectangular metamer of a spectral reflectance function the $a\delta\lambda$ color descriptors of the spectral reflectance function. The $a\delta\lambda$ color descriptors will be used to represent spectral reflectance functions in a 3D space (Object-color space section).

**Perceptual correlates of the $a\delta\lambda$ color descriptors**

The $a\delta\lambda$ color descriptors for the spectral reflectance functions of 1600 Munsell papers (retrieved from http://spectral.joensuu.fi/databases) were evaluated for the CIE illuminant D65 using the cone fundamentals based on the photopigment spectra (Figure 3). The papers having the same spectral waveform and differing only in purity were found to have similar color appearance. Specifically, they differ in a perceptual dimension, which will be referred to as apparent purity. Intuitively, apparent purity indicates the strength of the chromatic quality (specified by the spectral waveform of the rectangular metamer). It should be mentioned, however, that apparent purity is different from Munsell Chroma. Indeed, having zero Munsell Chroma, all achromatic Munsell papers, nevertheless, differ in purity, and thus in apparent purity. For example, the purity of the gray paper with the spectral reflectance $x_{0.5}(\lambda)$ is zero whereas that of the white paper ($x(\lambda) \equiv 1$) is maximal (i.e., equal to 1). The apparent difference between various spectral reflectances that have the same purity will be referred to as that in chromaticity. In other words, the chromatic base specifies chromaticity of object-color stimuli.

It was found that the papers of the same Munsell Hue had rather close central wavelengths. This tendency is illustrated by Figure 12 where the papers with maximal chroma (for each value) from five pages of the Munsell book are displayed. When the papers of the same Munsell Hue are of the same type, they are arranged in nearly vertical rows that are well separated and ordered in the horizontal direction. However, there is a considerable misalignment between the papers of the same Munsell Hue of different type. For example, three blue and three purple papers of type II are shifted clockwise in the visible spectrum circle relative to the blue and purple papers of type I. One yellow and two red papers of type II are shifted counterclockwise. This misalignment will be discussed below (see Wavelength reparameterization section).

While the symbols of the same color are also ordered in the vertical direction in Figure 12 (the larger Munsell Value, the broader the spectral bandwidth), there seems to

![Figure 11](image1.png) **Figure 11.** Regular object-color solid same as in Figure 8 except that the coordinate lines are based on spectral band and central wavelength.

![Figure 12](image2.png) **Figure 12.** Color descriptors of the Munsell papers of maximal chroma from five pages (10B, 10G, 10Y, 10R, and 10PB) after wavelength reparameterizing. Each paper is plotted as a point with the Cartesian coordinates $\delta$ (ordinate) and $\lambda$ (abscissae). The color of the symbols resembles the hue in the page. The red and blue lines represent the main and opposite meridians, respectively. A triangle made by these lines and the horizontal axis represents the reflectance spectra of type I. The points above the blue and red lines represent those of type II.
be no simple relationship between spectral bandwidth and Munsell Value.

Alternatively, it is the strength of the white and black hue components (referred to as blackness and whiteness) that seems to correlate with the spectral bandwidth. Indeed, whiteness was found to increase and blackness to decrease with the spectral bandwidth. Consider, for example, a series of Munsell papers of the same central wavelength. When the spectral bandwidth is close to zero, the papers look black. When the spectral bandwidth gradually increases, the papers get tinged with some chromatic hue, losing their blackness. For some value of spectral bandwidth, the hue becomes purely chromatic (containing no blackness at all). Further increase of spectral bandwidth results in a tinge of whiteness. Whiteness becomes maximal for the spectral bandwidth value close to \( (\lambda_{\text{max}} - \lambda_{\text{min}}) \).

**Wavelength reparameterization**

A large shift in central wavelength between the Munsell papers of the same Munsell Hue but different type (Figure 12) can be, at least partly, due to that the ends of the visible spectrum, \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \), have been inappropriately defined. Note that the length of the visible spectrum interval \( (\lambda_{\text{max}} - \lambda_{\text{min}}) \) is involved in the definition of both spectral bandwidth and central wavelength for spectral reflectance functions of type II (Equations 13 and 14). Any change of \( \lambda_{\text{min}} \) and/or \( \lambda_{\text{max}} \) will affect the \( a \delta \lambda \) color descriptor values. Therefore, if, say, the long wavelength end, \( \lambda_{\text{max}} \), is unduly in excess, this will result in a central wavelength shift for spectral reflectance functions of type II similar to that observed in Figure 12.

It seems natural, then, to define more optimally the visible spectrum interval, leaving outside it the wavelengths not contributing into producing color signal. Figure 13 presents the length of the opposite meridian of the regular object-color solid (the blue coordinate line in Figure 8) as a function of wavelength. As one can see, the length practically does not differ from zero for \( \lambda < 405 \) nm and reaches its maximum approximately at 650 nm. To be more exact, the length of the fragments of the opposite meridian for \( 380 \leq \lambda \leq 405 \) is less than 0.01 and more than 0.99 for \( 380 \leq \lambda \leq 650 \). Therefore, given a 1% accuracy, the values 405 and 650 nm can be taken as effective ends of the visible spectrum interval.

Alternatively, a rather drastic solution will be to reparameterize the visible spectrum interval so as to make the wavelength contribution into producing the color signal more uniform. For instance, one can take length of the meridian in the regular object-color solid as a new parameter. The length \( \sigma \) of the arc of the opposite meridian between the points \( (s_1(\lambda_{\text{min}}), s_2(\lambda_{\text{min}}), s_3(\lambda_{\text{min}})) \) and \( (s_1(\lambda), s_2(\lambda), s_3(\lambda)) \) \( (\lambda_{\text{min}} \leq \lambda \leq \lambda_{\text{max}}) \) is given by

\[
\sigma(\lambda) = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} \sqrt{(s_1(\mu))^2 + (s_2(\mu))^2 + (s_3(\mu))^2} \, d\mu.
\]  

(17)

Letting \( \overline{\sigma}(\lambda) = \sigma(\lambda)/\sigma(\lambda_{\text{max}}) \), define a parameter \( \omega = \sigma(\lambda) \). Note that as \( \sigma(\lambda_{\text{min}}) = 0 \), and \( \sigma(\lambda_{\text{max}}) = 1 \), \( \omega \) varies from 0 to 1. Figure 14 presents the same cone fundamentals as in Figure 3 in terms of the reparameterized wavelength, \( \omega \).

The spectral bandwidth and central wavelength of rectangular spectral reflectance functions can be redefined in terms of the new parameter \( \omega \). Specifically, given

![Figure 14](http://jov.arvojournals.org/pdfaccess.ashx?url=/data/journals/jov/933556/)
a rectangular spectral reflectance function \( x_2(\lambda; \lambda_1, \lambda_2) \) (Equation 5), define

\[
\tilde{\delta} = \begin{cases} 
|\omega_1 - \omega_2|, & \text{for } x_2(\lambda; \lambda_1, \lambda_2); \\
1 - |\omega_1 - \omega_2|, & \text{for } 1 - x_2(\lambda; \lambda_1, \lambda_2); 
\end{cases}
\]

where \( \omega_i = \sigma(\lambda_i) \) \( (i = 1, 2) \). Similarly, define

\[
\tilde{\lambda} = \begin{cases} 
0.5(\omega_1 + \omega_2), & \text{for } x_2(\lambda; \lambda_1, \lambda_2); \\
0.5((\omega_1 + \omega_2) + 1), & \text{for } 1 - x_2(\lambda; \lambda_1, \lambda_2) \text{ if } (\omega_1 + \omega_2) < 1; \\
0.5((\omega_1 + \omega_2) - 1), & \text{for } 1 - x_2(\lambda; \lambda_1, \lambda_2) \text{ if } (\omega_1 + \omega_2) > 1. 
\end{cases}
\]

In Figure 15, the same object-color solid as in Figure 8 is presented with the coordinate lines equally spaced in terms of the meridian length parameter \( \omega \). The space between the coordinate lines in Figure 15 corresponds to 1/30. As one can see, the coordinate web is uniform now.

The color descriptors \( \alpha, \delta, \) and \( \lambda \) lend themselves to be interpreted as coordinates. Dropping off purity \( \alpha \), one can use spectral bandwidth and central wavelength, \( \delta \) and \( \lambda \), as the Cartesian coordinates to represent a spectral reflectance as a point in a plane. Such a representation will be referred to as the \( \delta \lambda \) diagram. It shows which element of the rectangle chromatic base should be taken to produce (in the sense of Equation 10) the rectangular metamer of the spectral reflectance in question. Figure 16 presents in the \( \delta \lambda \) diagram the same Munsell papers as Figure 12. Note that, unlike Figure 12, there is almost no misalignment between the papers of different type.

The \( a\delta \lambda \) color descriptors of the Munsell and NCS papers

Figure 17 displays all the Munsell papers in the \( \delta \lambda \) diagram. As one can see, the Munsell papers are rather unevenly distributed. There are large “lacunas” void of the Munsell papers. The Munsell collection is believed to be uniform. Specifically, adjacent papers in the Munsell tree are supposed to be separated from each other by small, presumably, equal perceptual distances. It follows that even after wavelength reparameterizing the \( \delta \lambda \) diagram remains perceptually non-uniform.

As many symbols in Figure 17 overlap, the histograms in Figures 18–20 give better understanding how the Munsell papers are distributed along each of the \( a\delta \lambda \) dimensions. Figure 18 shows that purity gravitates to values more than 0.5. Most of the Munsell papers have the spectral

Figure 15. Regular object-color solid with the coordinate lines in term of reparameterized wavelength \( \omega \).

Figure 16. Color descriptors of the Munsell papers of maximal chroma from five pages (10B, 10G, 10Y, 10R, and 10PB) after wavelength reparameterizing. Each paper is plotted as a point with the Cartesian coordinates \( \tilde{\delta} \) (ordinate) and \( \tilde{\lambda} \) (abscissae).

Figure 17. The distribution of 1600 Munsell papers in the \( \delta \lambda \) diagram. Axes as in Figure 16. The papers of the same Munsell Hue are depicted by symbols of the same color. The papers of the same Munsell Value are represented by symbols of the same shape. The red and blue lines represent the main and opposite meridians, respectively.
bandwidth value less than 0.25 (Figure 19). Central wavelength is also distributed rather unevenly (Figure 20). There are four well-pronounced peaks—around the \( \lambda \) values 0, 0.4, 0.75, and 1, that correspond to the ends of the visible spectrum interval and the wavelengths 475 and 575 nm. Interestingly, the value \( \lambda = 0.4 \) is the one at which all the three curves in Figure 14 have approximately equal ordinates.

To find out whether such uneven distribution in the \( \delta \lambda \) diagram is a particular feature of the Munsell collection, the \( \delta \alpha \lambda \) color descriptors have been also evaluated for the 1950 NCS papers (Hard & Sivik, 1981) for the same illuminant and cone fundamentals. Figure 21 presents the Munsell and NCS papers in the \( \delta \lambda \) diagram. Both the Munsell and NCS collections are distributed remarkably similarly.

Presenting both Munsell and NCS papers in the same \( \delta \lambda \) diagram allows one to see how these two different color systems relate to each other. Moreover, this allows for transforming the Munsell notations into that of NCS and vice versa, giving us a way to go between the system. The practical importance of such a transformation is generally acknowledged (e.g., Nayatani, 2004).

**Object-color space**

The \( \delta \lambda \) diagram has two major shortcomings. Firstly, it does not represent purity. For purity to be represented, one needs one more spatial dimension. Secondly, mapping the rectangle chromatic base onto a rectangle area in the Cartesian plane is incorrect from the topological point of view. It is discontinuous at both the perfect reflector \( (x(\lambda) \equiv 1) \) and the ideal black \( (x(\lambda) \equiv 0) \). Indeed, for any central wavelength, a rectangular spectral reflectance function converges to \( x(\lambda) \equiv 0 \) when the spectral bandwidth goes to zero. However, a point \( (\delta_0, \lambda_0) \) in the \( \delta \lambda \) diagram...
will converge to \((0, \lambda_0)\) when \(\delta_0 \to 0\). Thus, the points with different \(\lambda_0\) will converge to different points in the bottom side of the rectangle area in Figure 21. One can avoid such a break of continuity mapping the rectangle chromatic base onto a sphere as described below.

**Geographical coordinates for spectral reflectance functions**

Using the geographical coordinates, one can represent a rectangle color atlas as follows. A rectangular metamer with purity \(a\), spectral bandwidth \(\delta\), and central wavelength \(\lambda\) is represented as a point in a 3D space with the Cartesian coordinate system such that the distance \(\rho\) of the point from the origin equals the purity, i.e.,

\[
\rho = a, 
\]

the geographical latitude \(\beta\) and longitude \(\theta\) being

\[
\beta = \pi \delta - \pi / 2; \quad \theta = 2 \pi \lambda. 
\]

The 3D space endowed with the geographical coordinate system such that the coordinates are retrieved from the \(a\delta\lambda\) color descriptors by Equations 20 and 21 will be referred to as the \(a\delta\lambda\) object-color space.

The equator (as well as any circle parallel to it) represents the visible spectrum circle. The meridian semicircles are loci of constant central wavelength. Therefore, longitude encodes central wavelength. A circle parallel to the equator is the loci of constant spectral bandwidth. Thus, latitude encodes spectral bandwidth. The north and south poles represent \(x(\lambda) \equiv 1\) (i.e., the perfect reflector) and \(x(\lambda) \equiv 0\) (i.e., the ideal black), respectively. Spectral bandwidth and central wavelength serve as the internal coordinates on the spherical surface \(\rho = 1\); thus, every rectangular spectral reflectance function can be uniquely located on it. Therefore, the rectangle chromatic base maps onto the unit sphere dropping down the information of their purity (Figures 22 and 23). As in Figure 17, symbol shape encodes Munsell Value and symbol color Munsell Hue. In other words, all the Munsell papers were projected on the unit sphere dropping down the information of their purity (Figures 22 and 23). In contrast with the real object-color solid that changes its shape as a result of the linear transformation of the cone fundamentals, the \(a\delta\lambda\) object-color solid is invariant with respect to any non-singular linear transformation of the cone fundamentals. Furthermore, although it is not invariant relative to an illuminant change, it remains almost the same for any illuminant that takes non-zero value across the visible spectrum (see Optimal metamers and optimal color atlas section).

**Spherical representation of the Munsell papers**

As its maximum deviation from the unit ball (in purity unit) does not exceed 0.01, the \(a\delta\lambda\) object-color solid for the cone fundamentals based on the photopigment spectra (Figure 3) is practically indistinguishable from the unit ball (Figures 22 and 23). Since the Munsell papers have purity of the magnitude less than 1 (Figure 18), they all lie inside of the unit ball. For each Munsell paper, a point on the \(a\delta\lambda\) object-color solid boundary (i.e., the chromaticity diagram) was determined, which lies on the same radius. In other words, all the Munsell papers were projected on the unit sphere dropping down the information of their purity (Figures 22 and 23). As in Figure 17, symbol shape encodes Munsell Value and symbol color Munsell Hue. In spite of the metric difference between the \(\delta\lambda\) diagram and the spherical chromaticity diagram, Munsell papers are also quite unevenly distributed over the spherical surface in Figures 22 and 23. As can be seen in Figure 23,
Munsell papers gravitate toward the south pole of the sphere. Still, there are vast areas on the sphere void of Munsell papers.

**Illuminant- and observer-induced color stimulus shifts**

**Illuminant-induced color stimulus shift**

When illumination changes an object-color stimulus can move from one class of metamericism to another. Hence, the color coordinates of a reflecting object may alter when the illumination changes because of the shift of its rectangular metamer from one metameric class to the other. Such a shift of an object-color stimulus over classes of metamerism induced by a change in illumination (referred to as the *illuminant-induced color stimulus shift*) results in an alteration of object color. For instance, as mentioned in the Optimal metamer and optimal color atlas section, object-color stimulus having complementary colors may get non-complementary colors because of the illuminant-induced color stimulus shift.

The illuminant-induced color stimulus shift caused by a replacement of the CIE illuminant D65 with the CIE illuminant A has been examined for Munsell and NCS papers. Specifically, using Equation 12 the $a\delta\lambda$ color descriptors of 1600 Munsell papers were calculated for the illuminants D65 and A using the cone fundamentals depicted in Figure 3. Figure 24 shows how purity evaluated for the CIE illuminant A covaries with that evaluated for the CIE illuminant D65. Likewise, Figures 25 and 26 present the spectral bandwidths and central wavelengths as evaluated for the CIE illuminants D65 and A, respectively.

For the purpose of its quantification, an illuminant-induced color stimulus shift can be decomposed into three component shifts. Specifically, let $a$, $\delta$, and $\lambda$ be the purity, spectral bandwidth, and central wavelength of the rectangular metamer for a spectral reflectance $x(\lambda)$ under one illumination, and $a'$, $\delta'$, $\lambda'$ under the other. Then the illuminant-induced purity shift for $x(\lambda)$ can be quantified by the difference $a - a'$, the illuminant-induced spectral bandwidth shift by $\delta - \delta'$ and the illuminant-induced central wavelength shift by $\lambda - \lambda'$.
The component color stimulus shifts produced by switching over from the illuminant \( D_{65} \) to \( A \) have been derived from the data presented in Figures 24–26. Specifically, the color descriptors \( \alpha, \delta, \lambda \) have been calculated for the CIE illuminant \( D_{65} \) and then \( \alpha', \delta', \lambda' \) for the CIE illuminant \( A \). The distribution of the purity shift magnitude, \( |\alpha - \alpha'| \), is given in Figure 27. Purity of the Munsell collection turns out to be rather robust to the illuminant change. Specifically, it has been found that 1023 Munsell papers (64%) change their purity by 0.01 or less, 1404 papers (88%) changing it by not more than 0.025. Similarly, Figures 28 and 29 present the distributions of the spectral bandwidth shift magnitude, \( |\delta - \delta'| \), and central wavelength shift magnitude, \( |\lambda - \lambda'| \), respectively.

Both the spectral bandwidth and central wavelength shifts contribute into the shift in chromaticity. It must be said, however, that the latter quantity is rather ambiguous. Indeed, a shift of the same magnitude \( |\lambda - \lambda'| \) along the equator might result in a larger chromaticity difference than along circles closer to the poles. In other words, the distance between two points on the chromaticity sphere separated by the same difference \( (\lambda - \lambda') \) depends on their latitudes (i.e., \( \delta \) and \( \delta' \)).

A more appropriate index of the chromaticity shift would then be a distance on the chromaticity sphere between two points determined by \((\delta, \lambda)\) and \((\delta', \lambda')\). As known, the distance between two points on a sphere is measured by the length of the shorter arc of the great

Figure 26. Effect of illuminant on central wavelength (CIE \( D_{65} \) vs. CIE \( A \)). Each of the 1600 points represents the central wavelengths, \( \lambda \), of a Munsell paper as evaluated for \( D_{65} \) (abscissae) and \( A \) (ordinate).

Figure 27. Purity shift \( |\alpha - \alpha'| \) for Munsell papers induced by the illuminant shift from \( D_{65} \) to \( A \).

Figure 28. Spectral bandwidth shift \( |\delta - \delta'| \) for Munsell papers induced by the illuminant shift from \( D_{65} \) to \( A \).

Figure 29. Central wavelength shift \( |\lambda - \lambda'| \) for Munsell papers induced by the illuminant shift from \( D_{65} \) to \( A \).
For large purity differences, Equation 23 is not appropriate. Indeed, consider, for example, two stimuli lying on the diameter of the equatorial circle of the CIE color solid (i.e., \( \beta = \beta' = 0 \) and \( \theta - \theta' = \pi \)): one of full purity (\( c = 1 \)) and the other with the purity close to zero, that is, \( c' \approx 0 \). In this case, Equation 23 yields \( D \approx c/2 \). Therefore, despite that these object-color stimuli differ practically only in purity, the Equation 23 gives a measure substantially different from zero. To avoid such a “pathology,” Equation 23 can be modified as follows

\[
D = \frac{\min\{c, c'\}}{\pi} \arccos (\cos \beta \cos \beta' \cos (\theta - \theta') + \sin \beta \sin \beta').
\] (24)

When \( c - c' \) is small, both Equations 23 and 24 give approximately the same value. However, when \( c' \to 0 \), \( D \) in Equation 24 approaches zero for any \((\delta, \lambda)\) and \((\delta', \lambda')\).

Using Equation 24, the chromaticity shift induced by the illuminant shift (from D65 to A) has been evaluated for 1600 Munsell papers (Figure 30) and 1950 NCS papers (Figure 31). The mean chromaticity shift for 1600 Munsell papers is 0.0118; the maximum chromaticity shift being 0.1283. For 57.6% of the Munsell papers, the chromaticity shift does not exceed 0.01, for 83% 0.02, and for 92.3% 0.03. A similar result has been observed for NCS papers (Figure 31). Specifically, the mean and maximum chromaticity shifts are found to be 0.0135 and 0.1349, respectively. Nearly half the NCS papers (48.5%) undergo chromaticity shift of not more than 0.01, 79.1% changing their chromaticity by not more than 0.02, and 92% not more than 0.03.

It should be borne in mind that the chromaticity difference (Equation 24) is an arbitrary stimulus measure.

For one can see, the distance \( d \) is proportional to purity, \( c \). This is in line with our intuition that the chromaticity difference between the object-color stimuli decreases with their purity. Equation 22 will be used to quantify the chromaticity difference between the object-color stimuli with color descriptors \((c, \delta, \lambda)\) and \((c', \delta', \lambda')\).

This formula can also be used for the object-color stimuli of different purities, \( c \) and \( c' \), provided that the purity difference \( |c - c'| \) is not large. In this case, the average purity, \((c + c')/2\), can be put as \( c \) in Equation 22. More specifically, to measure the chromaticity difference between the object-color stimuli with color descriptors \((c, \delta, \lambda)\) and \((c', \delta', \lambda')\) when \( c - c' \) is small the following formula can be used:

\[
D = \frac{(c + c')}{2\pi} \arccos (\cos \beta \cos \beta' \cos (\theta - \theta') + \sin \beta \sin \beta').
\] (23)

Due to normalizing by a factor \( \pi \) in Equation 23, the maximum chromaticity difference equals 1.
In order to relate it to subjective color difference, the chromaticity differences (Equation 24) between adjacent Munsell Hues have been evaluated. More specifically, 40 Munsell papers—one from each page—have been selected so that each selected paper has maximal Munsell Chroma. Chromaticity differences (Equation 24) have been evaluated for 40 consecutive pairs in this selection (Figure 32). Maximal difference, 0.145, has been found between the papers 10Y8.5/12 and 2.5GY7/12, the minimal, 0.007, between 5RP5/12 and 7.5RP5/14. The mean chromaticity difference is found to be 0.0366, the median being 0.0250. Therefore, the mean chromaticity difference between adjacent Munsell Hues is more than three times as much as the mean chromaticity shift induced by the illuminant shift (D65 to A) observed for all the 1600 Munsell papers. Hence, on average, a change in color for Munsell papers induced by the shift from the illuminants D65 to A is, by and large, at the limit of our ability to discern chromatic differences.

The main motivation for developing a new color space has been that the CIE 1931 color space, and its derivatives can be used to represent the color of objects only if the illumination of these objects is the same because when the illumination changes the vector representation (i.e., tristimulus values) changes as well. As one can see, the color coordinates of a reflecting object in the a*b* object-color space vary with illumination as well (because of illuminant-induced color stimulus shift). Yet, there is an important difference between these two cases. In the case of the CIE 1931 color stimulus representation, the coordinate system itself changes with illumination. For example, a sheet of paper under (i) the direct sun light and (ii) the light from blue sky will reflect different lights. These lights will be represented by two different points in the CIE 1931 color space. This difference is, generally, produced by both a possible change of the sheet color (due to the illuminant-induced color stimulus shift) and by the change in the coordinate system. In the a*b* object color, these two contributions are well separated because the coordinate system in this space does not depend on illuminant. That is, the new space is invariant of illumination change. Moreover, any change of the coordinates of a reflecting object evaluated for two illuminants indicates the change of the object color induced by the illuminant alteration. Hence, the new space can be used to predict the effect of the illumination on object color that is hard to achieve with the CIE 1931 color space and its derivatives.

Observer-induced color stimulus shift

As a given spectral reflectance function can be metameric to different rectangle metamers for different observers (i.e., different sets of cone fundamentals), the color coordinates of the same object can be different for different observers. A chromaticity shift similar to an illuminant-induced color stimulus shift can occur due to a change of cone fundamentals. In attempt to ascertain how robust to alteration of cone fundamentals chromaticity is, the chromaticity shift induced by replacing the cone fundamentals depicted in Figure 3 with Smith and Pokorny’s cone fundamentals (Smith & Pokorny, 1975) for 1600 Munsell papers has been evaluated (Figure 33). Only 5 of 1600 papers undergo the chromaticity shift exceeding 0.03, the chromaticity shift less than 0.02 taking place for 88% papers. The mean chromaticity shifts induced by the cone fundamentals change is found to be 0.0057 that is less than the minimal chromaticity difference between Munsell Hues (0.007). Generally, 57% Munsell papers exhibit a chromaticity shift less than
0.007. Therefore, at least half the papers undergo a chromaticity shift (due to the cone fundamentals change), which is most unlikely to be noticed by a human observer.

### Discussion

A spatial representation of object color implies a sample of spectral reflectance functions, i.e., color atlas (Koenderink, 1987; Koenderink & Doorn, 2003). It is the color atlas that undergoes a spatial order whereby all the colors it renders also get spatially arranged. Ideally, a color atlas (thus, the color order system based on it) is to specify a spectral reflectance in terms of some perceptual color dimensions. In fact, the color order systems available at the present are rather far from this ideal. Indeed, the spatial order is usually imposed on the color atlas either in terms of some perceptual (e.g., Munsell atlas, NCS) or physical dimensions (e.g., Koenderink & Doorn, 2003; Ostwald, 1931). Being based on intuitively clear perceptual dimensions, the perceptual color order systems usually do not provide an easy way to characterize an arbitrary spectral reflectance in terms of these dimensions. For instance, 200 years ago Runge, with his color sphere, anticipated the geometrical form of the object-color manifold (Kuehni, 2003, pp. 59–62). However, one cannot locate a particular spectral reflectance in Runge’s sphere. Nor can one compute Munsell Hue, Chroma, and Value for an arbitrary spectral reflectance function. On the other hand, the dimensions of the physical color order systems do not lend themselves readily to perceptual interpretation. For example, representing a spectral reflectance as a weighted combination of three basis spectral reflectance functions, the linear models specify each arbitrary spectral reflectance by the three numbers (for a review, see Brainard, 1995; Hurlbert, 1998; Maloney, 2003). However, a phenomenological interpretation of these numbers is hardly possible.

An ideal solution would be a physical color order system whose dimensions allow straightforward phenomenological interpretation. Ostwald’s color system (Ostwald, 1931) can be considered as a significant step toward this ideal. This system is based on a set of rectangular spectral reflectance functions that differ from the rectangle color atlas put forward here in the following. Firstly, Ostwald’s rectangular spectral reflectance function takes two values, the central wavelength proves to characterize the hue of a rectangular metamer (thus, any spectral reflectance metameric to it) as accurately as the dominant wavelength does the hue of a monochromatic light. An important feature of the rectangle color atlas is that the color descriptors of the optimal rectangular spectral reflectance functions remain practically constant with respect to illumination. Particularly, the chromaticity coordinates and purity of the perfect reflector (λ(λ) = 1) and the ideal black (λ(λ) = 0) are the same for all the illuminants (with positive spectral power distribution). This is not the case when reparameterization of the optimal color atlas is based on a linear representation unless the basis spectral reflectance functions themselves include the perfect reflector or ideal black. Generally, the linear coordinates of the optimal rectangular spectral reflectance functions will change with illumination.

...
Furthermore, the basis spectral reflectance functions move from one class of metamerism to another when illumination alters. Therefore, the linear representation of a metamer class is performed in terms of the basis that is subject to uncontrollable change when illumination alters. In other words, the linear models do not provide a frame reference robust to illumination change. From this point of view, using the linear models for representing spectral reflectance provides no advantage to the common practice of using the CIE 1931 color space to represent object color.

Likewise, a color atlas based on Gaussian spectral reflectance functions (MacLeod & Golz, 2003; Nikolaev, 1985) does not provide a frame reference robust to illumination change either. Neither does any physical finite sample of spectral reflectance functions (e.g., Munsell atlas, NCS). Indeed, being subject to color stimulus shift, the Munsell papers may change their class of metamerism with illumination. Therefore, if some spectral reflectance remains metameric to the same Munsell paper under two different illuminants, this will not necessarily guarantee that its color remains the same. It might simply mean that this spectral reflectance changes its color. The same way as the Munsell paper in question.

Conclusion

The new color space is suitable for computational representation of the object colors encoded by not only humans but also artificial devices (digital cameras, LCD and CRT displays, color printers, etc.). Specifically, the proposed new color space can be used as a device-independent color space in color management systems for input (digital cameras, scanners) and output (displays, printers) device calibration and characterization. There are a number of advantages of the proposed color space to the CIE 1931 colorimetric space (as well as the sRGB color space and many others based on it) currently used as a device-independent color space (e.g., Bala, 2003). Firstly, the proposed color space is a 3D affine space in which any color stimulus is represented as a point with the coordinates $(\varphi_1(w), \varphi_2(w), \varphi_3(w))$. Secondly, the dimensions of the proposed color space have clear perceptual correlates. Thirdly, the proposed color space is independent of illumination. Furthermore, the new color space is robust to not only illuminant but observer as well.

Appendix A

Basic terms and definitions

Object-color stimulus: An object-color stimulus is a spectral reflectance.

Illuminant: An illuminant is spectral power distribution.

Remark: Object-color stimulus times illuminant amounts to color stimulus as defined by Wyszecki and Stiles (1982, p. 723).

Sensor: A sensor $\varphi$ is a linear device the response, $\varphi(w)$, of which to a color stimulus $w(\lambda)$ is given by

$$\varphi(w) = \int_{\lambda_{\min}}^{\lambda_{\max}} s(\lambda)w(\lambda)d\lambda, \quad \text{(A1)}$$

where $s(\lambda)$ is the spectral sensitivity of the sensor and $[\lambda_{\min}, \lambda_{\max}]$ is the visible spectrum interval.

Color signal: The color signal produced by sensors $\varphi_1$, $\varphi_2$, and $\varphi_3$ in response to a color stimulus $w(\lambda)$ is a triplet $(\varphi_1(w), \varphi_2(w), \varphi_3(w))$.

Color signal space: Given sensors $\varphi_1$, $\varphi_2$, and $\varphi_3$, a color signal space is a 3D affine space in which any color stimulus $w(\lambda)$ is represented as a point with the coordinates $(\varphi_1(w), \varphi_2(w), \varphi_3(w))$.

Metamer object-color stimuli: Two physically different object-color stimuli producing the same color signals under some illuminant are called metamer under this illuminant.

Object-color: An object-color is a class of metameric object-color stimuli.

Object-color solid: The object-color solid is a set of points in the color signal space produced by all possible object-color stimuli for a fixed illuminant.

Optimal object-color stimuli: An optimal object-color stimulus is a spectral reflectance the color signal of which lies on the boundary surface of the object-color solid.

Remark: If $x(\lambda)$ is optimal object-color stimulus, then $1 - x(\lambda)$ is optimal object-color stimulus as well.

Complementary optimal object-color stimuli: Given an optimal object-color stimulus $x(\lambda)$, the optimal object-color stimulus $1 - x(\lambda)$ is called complementary to $x(\lambda)$.

Step function: A step (spectral reflectance) function is a piecewise constant function taking only two values: 0 or 1.

Remark: Each optimal object-color stimulus is a step spectral reflectance function.

Transition wavelengths: Transition wavelengths of a step spectral reflectance function are those where transition between 1 and 0 occurs.

Rectangular function: A rectangular (spectral reflectance) function is a step function taking 1 on the interval $[\lambda_1, \lambda_2]$ ($\lambda_{\min} \leq \lambda_1 \leq \lambda_2 \leq \lambda_{\max}$) and 0 outside it.

Complete color atlas: A complete color atlas is a set of (not metameric) object-color stimuli such that each object-color stimulus is metameric to one of its element. Illuminant invariant color atlas: A complete color atlas is said to be invariant with respect to the family of illuminants if and only if (i) each class of metamerism contains exactly one element of the atlas under any illuminant in the family and (ii) any two elements of the
Optimal color atlas: The optimal color atlas is a set of spectral reflectance functions expressed as

\[(1 - a)x_{0.5}(\lambda) + ax_{\lambda}(\lambda_1, \lambda_2), \quad (A2)\]

where \(x_{0.5}(\lambda)\) is the spectral reflectance function taking 0.5 at every wavelength \(\lambda\) within the visible spectrum interval \([\lambda_{\text{min}}, \lambda_{\text{max}}]\) and \(x_{\lambda}(\lambda_1, \lambda_2)\) runs over the whole set of optimal object-color stimuli.

Remark: The optimal color atlas is complete and invariant with respect to illuminants with positive spectral power distribution.

Rectangular metamer: The rectangular metamer of an object-color stimulus \(x(\lambda)\) is a piecewise constant spectral reflectance function, metameric to \(x(\lambda)\), which is given by

\[(1 - a)x_{0.5}(\lambda) + ax_{\lambda}(\lambda; \lambda_1, \lambda_2), \quad (A3)\]

where \(x_{0.5}(\lambda)\) is defined as above (Equation A2); \(x_{\lambda}(\lambda; \lambda_1, \lambda_2)\) is the rectangular spectral reflectance function with the transition wavelengths \(\lambda_1 \leq \lambda_2\); \(a\) is a real number.

Proper and improper rectangular metamers: The rectangular metamer (Equation A3) is called proper (respectively, improper) if \(|a|\) in Equation A3 is such that \(|a| \leq 1\) (respectively, \(|a| > 1\)).

Rectangular color atlas: The rectangle color atlas is the set of the rectangular metamers for all the object-color stimuli.

Central wavelength: The central wavelength of a rectangular spectral reflectance function with the transition wavelengths \(\lambda_1\) and \(\lambda_2\) is given by \((\lambda_1 + \lambda_2) / 2\).

Chromatic amplitude: The parameter \(a\) of an element of the rectangle color atlas \((1 - a)x_{0.5}(\lambda) + ax_{\lambda}(\lambda; \lambda_1, \lambda_2)\) is the chromatic amplitude. The chromatic amplitude of an object-color stimulus \(x(\lambda)\) is the chromatic amplitude of its rectangular metamer.

Purity: The purity of an object-color stimulus is the absolute value of its chromatic amplitude.

Complementary object-color stimuli: Two object-color stimuli are complementary if their rectangle metamers differ only by sign of the chromatic amplitude \(a\).

Chromaticity coordinates: The chromaticity coordinates of an object-color stimulus \(x(\lambda)\) are the quantities \(\beta\) and \(\theta\), which are related to the spectral bandwidth and the central wavelength of \(x(\lambda)\) as \(\beta = \pi \delta - \pi / 2\) and \(\theta = 2 \pi \lambda\), where \(\delta\) and \(\lambda\) are defined by Equations 18 and 19.

Object-color space: Object-color space is a 3D space with the geographical coordinate system such that each object-color stimulus \(x(\lambda)\) is represented as a point at a distance from the origin equal to its purity, the latitude and longitude being equal to the chromaticity coordinates of \(x(\lambda)\).

Chromaticity difference: Chromaticity difference between object-color stimuli with the chromaticity coordinates \(\beta, \beta', \theta, \theta'\) and purities \(c, c'\) is given by Equation 24.

Illuminant invariance of optimal color atlas (proof)

Let \(x_1(\lambda)\) and \(x_2(\lambda)\) be two different spectral reflectance functions given by Equation 10, that is, \(x_1(\lambda) = (1 - a') x_{0.5}(\lambda) + ax_2(\lambda; \lambda_1, \lambda_2)\) and \(x_2(\lambda) = (1 - a'') x_{0.5}(\lambda) + ax_2(\lambda; \lambda_1, \lambda_2)\), where \(x_{opt}(\lambda)\) and \(x_{opt}''(\lambda)\) are optimal spectral reflectance functions (with respect to some set of cone fundamentals) such that \(0 < a', a'' < 1\), and either \(x_{opt}(\lambda) \neq x_{opt}''(\lambda)\) or \(a' \neq a''\), or both the inequalities hold true. As illuminants with positive spectral power distribution do not change the optimal spectral reflectance function set, for any such illuminant, \(x_{opt}(\lambda)\) and \(x_{opt}''(\lambda)\) will remain optimal. Given an arbitrary illuminant with positive spectral power distribution, let us denote \(x_{opt}'\) and \(x_{opt}''\) the points in the object-color solid boundary into which \(x_{opt}(\lambda)\) and \(x_{opt}''(\lambda)\) map. Let \(x_{0.5}, x_{1}\), and \(x_{2}\) be the points in the object-color solid into which \(x_{0.5}(\lambda), x_{1}(\lambda)\), and \(x_{2}(\lambda)\) map, respectively. Assume that \(x_{1} = x_{2}\), that is, \((1 - a') x_{0.5} + ax_{opt}' = (1 - a'') x_{0.5} + ax_{opt}''\). Then, it implies that

\[\alpha x_{opt}' - \alpha x_{opt}'' = (\alpha' - \alpha'') x_{0.5}. \quad (A4)\]

If \(a' = a''\), then it follows from Equation A4 that \(x_{opt}' = x_{opt}'\). As there is no metamerism in the object-color solid boundary, \(x_{opt} = x_{opt}''\) implies \(x_{opt}'(\lambda) = x_{opt}''(\lambda)\) that is impossible because \(x_{1}(\lambda)\) and \(x_{2}(\lambda)\) are different. Therefore, \(a' \neq a''\).

Assume for certainty that \(a' > a''\) and rewrite Equation A4 as

\[x_{opt}' = \frac{a''}{a'} x_{opt}'' + \left(1 - \frac{a''}{a'}\right) x_{0.5}. \quad (A5)\]

As \(0 < \frac{a''}{a'} < 1\), it follows from Equation A5 that \(x_{opt}'\) is an interior point of the object-color solid that is impossible because \(x_{opt}'(\lambda)\) is optimal.
Therefore, \( \bar{x}_1 \) and \( \bar{x}_2 \) are different points in the object-color solid. So we conclude that different elements \( x_1(\lambda) \) and \( x_2(\lambda) \) of the optimal color atlas cannot become metameric under illuminants with positive spectral power distribution.

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**Footnotes**

1 Although this intuitively clear statement has been made by several authors before (e.g., Ostwald, 1923; Wyszecki & Stiles, 1982; although see Maximov, 1984, p. 73), the formal proof has not been known until recently (Logvinenko & Levin, 2009).

2 See Koenderink (1987) for a notion of color atlas.

3 Note that the symmetrical location of complementary optimal metamers in the object-color solid follows from the definition of complementarity rather than underlies it.

4 The distance between Munsell papers in the plane \( \delta \lambda \) diagram is different from that in the spherical chromaticity diagram.

5 A great circle is that whose center is coincident with the center of the sphere.

6 At best, it can be done only on the ground of visual (subjective) inspection provided the spectral reflectance is implemented as a real surface.

7 Needless to say that at the time, no experimental data corroborated Runge’s guess. Yet, more recently, it was supported by multidimensional analysis of Munsell papers (Izmailov, 1995).

8 Strictly speaking, it is only Albert Munsell himself who could do this provided that the reflectance function is implemented as a surface available for visual inspection.

9 It is a shift from one class of metamerism to another rather than a phenomenological alteration that is meant here by changing color.

**References**


